THE PHYSICAL SOCIETY OF LONDON.

PROCEEDINGS.

VOLUME XVI.—PART VI.

JUNE 1899.

LONDON:

TAYLOR AND FRANCIS, RED LION COURT, FLEET STREET.

1899.

PHYSICAL SOCIETY OF LONDON.

1899–1900.

OFFICERS AND COUNCIL.

President.

PROF. OLIVER J. LODGE, D.Sc., LL.D., F.R.S.

Vice-Presidents

WHO HAVE FILLED THE OFFICE OF PRESIDENT.

DR. J. H. GLADSTONE, F.R.S.
PROF. G. C. FOSTER, F.R.S.
PROF. W. G. ADAMS, M.A., F.R.S.
THE LORD KELVIN, D.C.L., LL.D., F.R.S.
PROF. R. B. CLIFTON, M.A., F.R.S.
PROF. A. W. REINOLD, M.A., F.R.S.
PROF. W. E. AYRTON, F.R.S.
PROF. G. F. FITZGERALD, M.A., F.R.S.
PROF. A. W. RÜCKER, M.A., D.Sc., Sec. R.S.
CAPT. W. DE W. ABNEY, R.E., C.B., D.C.L., F.R.S.
SHELFORD BIDWELL, M.A., LL.B., F.R.S.

Vice-Presidents.

T. H. BLAKESLEY, M.A.
C. VERNON BOYS, F.R.S.
G. GRIFFITH, M.A.
PROF. J. PERRY, D.Sc., F.R.S.

Secretaries.

H. M. ELDER, M.A.
50 City Road, E.C.

W. WATSON, B.Sc.
7 Upper Cheyne Row, Chelsea, S.W.

Foreign Secretary.

PROF. S. P. THOMPSON, D.Sc., F.R.S.

Treasurer.

DR. E. ATKINSON.
Portesbery Hill, Camberley, Surrey.

Librarian.

W. WATSON, B.Sc.
Royal College of Science, South Kensington.

Other Members of Council.

PROF. H. E. ARMSTRONG, D.Sc., F.R.S.
WALTER BAILY, M.A.
R. E. CROMPTON.
PROF. J. D. EVERETT, D.C.L., F.R.S.
PROF. A. GRAY, LL.D., F.R.S.
E. H. GRIFFITHS, M.A., F.R.S.
PROF. J. VIRIAMU JONES, M.A., F.R.S.
S. LUPTON, M.A.
PROF. G. M. MINCHIN, M.A., F.R.S.
J. WALKER, M.A.

XXXVII. *Presidential Address delivered by* Prof. OLIVER LODGE, *LL.D., D.Sc., F.R.S., at the Annual General Meeting on February* 10*th*, 1899.

PART I.—*Preliminary Portion.*

IN taking the Chair of this Society for one year, notwithstanding that I am a country member and therefore shall not be able always to be present, I am I daresay traversing the views of what is considered best for the Society by a minority of members. I have no knowledge that it is so, but it can hardly be otherwise. I hesitated myself for some time, but ultimately consented to be put in nomination by the Council; and I can promise that I shall try to be present as often as my duties permit.

I have been a member of the Society from its earliest infancy, for though not technically an "original member," nor one admitted in the first year, that is a mere accident of not having presented myself for admittance by the Chair; perhaps I had not paid my subscription, but I used to attend all the meetings, and my memory goes back to the preliminary discussions that preceded the birth of the Society, between Prof. Foster, Prof. Adams, Dr. Gladstone, Dr. Atkinson, and Dr. Guthrie, especially Dr. Guthrie, to whom the idea of the Society was, I believe, largely due.

In those days we used to meet at South Kensington on Saturday afternoons ; and my experience of recent years is altogether too remote and detached to make me aware whether the change, from the hospitable laboratory at South Kensington to the no doubt equally hospitable but less *physically* inspiring quarters of Burlington House, has been an unmixed advantage. If it is an unmixed advantage, the Society has been fortunate.

Our death-roll for the past year contains the names of two original members whose figures were familiar at the

meetings in those early days: Henry Perigal and Latimer Clark. It also contains the names of Sir J. N. Douglass, so well known in connexion with lighthouse work; of Dr. Obach, the skilled electrician and designer of Messrs. Siemens Bros.; of the Rev. Bartholomew Price, known not only as a mathematician but also as a man of public spirit and enterprize in the University of Oxford, who had I believe more to do with the early financial success of the Clarendon Press than is generally known; of Dr. J. E. Myers, and alas of Dr. John Hopkinson.

It is unfitting that this year's deliverance from the Chair of any Society of which John Hopkinson was a member should fail to contain some reference to his untimely loss; yet there is nothing for me to say but what you already feel, and no words of mine are needed. I cannot say that I knew him in his youth, but I heard of him. He and I as self-taught students—at least, I was at that time self-taught, and he could hardly have found a better teacher than himself—once figured together in the so-called honours list of the South Kensington May examinations in Electricity and Magnetism or else in Sound Light and Heat, or in both. Needless to say, his name stood above mine. He passed, for fun, in nearly the whole range of South Kensington subjects; I took a good many of them, but had nothing like either his comprehensive range or his educational advantages. And then he went and wrested the Senior Wranglership from that exceptionally brilliant mathematical genius, J. W. L. Glaisher. That was a surprising achievement for a Whitworth Scholar and practical Engineer. May I be allowed to congratulate Mrs. Hopkinson and her family not only on a peculiarly noble donation, but also on its special application to the welding together the names of Cambridge, of Engineering, and of Hopkinson.

It is remarkable, as Mr. Arthur Balfour said in a speech the other day (at the opening of the Battersea Institute, I think), it is remarkable how in modern times the pursuit of science tends in the direction of rapid and immediate application.

In Newton's time pure science was altogether aloof from

practice: I do not suppose that the gravitational theory of astronomy had any effect on His Majesty's coins at the Mint, nor did the prismatic sifting of white-light lead at that time to any chemical applications of spectrum analysis. Even as late as Faraday's day the induction of currents by motion and magnetism exercised no appreciable effect on commerce and industry.

Nowadays, Fourier's theorem is applied by Lord Kelvin to work an Atlantic cable; Hopkinson's researches on the magnetic circuit (an idea that has revolutionized the practical treatment of magnetism) result at once in improvements in the dynamo; and Hertz's experimental detection of Maxwell's waves seems likely to result before long in a new system of wireless telegraphy.

What is the cause of this keen interest in practical applications felt in varying degrees by nearly all modern physicists, felt most keenly and consistently I think, of those I know, by your past-President Prof. Ayrton? Is it the example and inspiration of Lord Kelvin, or is it that the human race generally is beginning to be better educated than it used to be, to take more kindly to the result of scientific researches, to be more eager to learn about them and, if possible, to apply them; partly, no doubt, because it has gradually discovered that by judicious treatment it can convert them into substantial commodity?

Surely, on the whole, with some drawbacks, the result from the point of view of science is good. If science were really remote from all human use or interest, as some scoffers still endeavour to maintain it to be, would there not be a fear lest gradually the human race should get tired of it, should cease to encourage even by approval or applause those whose instinct or mission it might be to develop it? So that gradually a blight might settle down, and advanced scientific knowledge become as extinct as a dead language or a fossil genus. A few scholars here and there would take an interest in it, as they did in the Middle Ages, and it might perhaps for a time be instilled into youth at the point of a cane, as the dead languages not so long ago used to be; but unless a branch of learning enters in some way into the life and well-

being of a community, unless it has some real interest—a literary interest it may be, a commercial interest perhaps, a living human interest of some kind—there is fear that however artificially bolstered up it may be for a time, by those whose stock in trade it is and who know little else, it will ultimately be suffered to go by the board and become to all practical intents and purposes dead, buried, and extinct.

I said just now that I supposed the public as a whole were gradually getting better educated, but how miserably slow is the process and how lamentable the present result. I have no feud with orthodox secondary schoolmasters, indeed some of them are my very good friends, but is it not painful to be unable to speak to the average so-called educated Englishman (except a few here and there of course) on any scientific discovery without at once being met by a hopeless wall of ignorance.

Is it so in other countries? I hardly think so; I do not think it is so marked even among Scotchmen, at any rate not among Scotchmen who have had their Natural Philosophy year at the University. A year for training in the whole of science—it is not much. If scientific men were ignorant of letters in the same proportion they would hardly be able to read, except in words of one syllable, nor write, except with the laborious contortions of the village urchin; a year extended on the subjects of all the Arts would not carry one very far. And as to sums, how many men there are whose mathematical equipment is limited to the practice of compound addition and subtraction, and who shy if they see an algebraic symbol, even in a newspaper,—indeed, when they occur in newspapers they are often of a kind that deserve to be shyed at.

They learn languages, some people seem to enjoy learning foreign languages—though the multiplicity of them is an artificial arrangement which an international convention might alter if it chose,—but the exact and beautiful language in which a great part of science is and must be written is to the majority of men unknown. Are teachers wholly free from blame in respect of the dislike which many boys feel

for the most elementary mathematics, for problems which may be made exhilarating when properly presented? I believe they are not. If the teacher dislikes mathematics, the boys will hate it, and if they hate it when young they will love it but little better when they grow up and proceed to be teachers themselves. There is a law of geometrical progression or compound interest here, and it seems to have been at work for some time. I must not however be understood here to be speaking of really cultivated intelligences or authorities in any subject. A thorough training in any branch of knowledge may so stimulate and enlarge the faculties as to fit them for intelligent appreciation and criticism of many other branches. Some men of letters I know whose minds are copiously and essentially scientific, they have not the detailed knowledge of the professed worker in science—that would be impossible and unnecessary—but they appreciate and assimilate all the main doctrines of science, and their broad criticisms and suggestions are often of value to the special worker. Any educational strictures on which I have ventured have reference not to scholars but to average people; at the same time I believe many scholars feel their deficiency in mathematics, and I believe it to be the result of bad teaching.

This is not my address, it is a preliminary excursus, my address has reference to a physical problem which has interested me, and about which I thought it might be interesting if I said something. It is customary to review some topic in an address, not to adduce anything specially new, but to treat of or attempt to clarify something already known. Usually it is some experimental fact which is thus treated, sometimes it is a point of theory; it is the latter to which I wish to ask your attention.

As to experiments, if I quickly review some of those that strike me in the past year—there is further progress to be reported concerning the developments of Prof. Zeeman's great discovery, the most important being (1) the new manifestations of it, or of a closely allied phenomenon, discovered by Professor Righi, to which Prof. S. P. Thompson called the attention of Section A at Bristol last September; and

(2) some extension of the resolution of complex lines, discriminating between the effect as exhibited by different lines, a discrimination which seems likely to lead to an extension of the powers of spectrum analysis by the application of a magnetic field to the source of light, as studied with apparatus of splendid dispersive power by Prof. Preston, and with his own ingenious method by Prof. Michelson. There is a notable continuation of the remarkably powerful and beautiful researches into atomic properties carried on at Cambridge by Prof. J. J. Thomson ; about which a great deal more might be said. And then there is the prediction made in the columns of 'Nature' by Prof. G. F. FitzGerald, but not yet verified, that circularly polarized light sent through an absorbing medium would constitute a magnet.

Many have been the attempts, from Faraday downwards, to excite magnetization by means of light, to detect a real and not an imaginary "magnetization of a ray of light," but all the experimenters hitherto have failed to realise the necessity of an absorbent medium. The effect is bound to be small, and of course it may not be there after all, but I understand it is being looked for, and it has at any rate been on rational grounds predicted.

Dr. Barton has continued experiments on the damping of waves on wires, and has not yet reconciled a discrepancy between theory and experiment ; being able, as I understand him, to account for only 5½ per cent. of the outstanding discrepancy. I have not myself looked into the matter, but I understand that a note has been communicated by Mr. Heaviside. I hope it will be published in full.

The subject of Terrestrial Magnetism, in the hands of Prof. Schuster on the theoretical side and of Prof. Rücker on the practical side, has recently entered on an enlarged existence and seems to be taking up a position almost of an independent science, so that when the multifurcation or explosion of Section A occurs, as sooner or later I fear it must, the fragment entitled Meteorology and Terrestrial Magnetism will perhaps not be one of the smallest.

The publication called 'Science Abstracts' has continued to be welcomed by workers all over the country, and the abstracts sent by Mr. Fournier d'Albe to the 'Electrician' are also useful and admirably done. I have no knowledge myself of the mode of control of the publication of our 'Science Abstracts,' but I confess to a personal preference for the older form, the one with a less startling cover, and with a list of authors' names, instead of unnecessary advertisements of well-known firms, on its back page. Undoubtedly the inauguration of these 'Abstracts' supplies a long-felt want, and even if one does not always find time to read them it is a comfort to feel that there they are, to be read in some less pressing season; and one need not envy the German worker his 'Beiblätter' as one used to.

But there is one most important event that has occurred during the year, perhaps to us conjointly as physicists the most important of all, an event of which the science of Physics will feel the effects—I trust the wholly favourable effects—for centuries to come: I mean the decision of the Government to begin on a small scale the inauguration of a National Physical Laboratory. If its effects are ever in any respects unfavourable to true science, a heavy indictment will lie against its future governing body, and against its Superintendent or Physicist Royal.

I wish to congratulate Sir Douglas Galton, as well as to some extent myself, on the result, the speedy result, of our urging of the subject, within the present decade, on the British Association; and as a Physical Society I think we owe a debt of gratitude to the Committee appointed by the Treasury to examine into the question, whose members gave themselves an immense amount of trouble, travelling to Berlin and elsewhere to make themselves acquainted with foreign ideals, and altogether going into the subject with extreme thoroughness, scientific judgment, and public spirit. The chairman of the committee was Lord Rayleigh, but during his absence in India the acting chairman was constantly Professor Rücker; Mr. Chalmers represented the Treasury with urbane frugality; and to Prof. Rücker and

Mr. Chalmers, as well as to the other members of the committee, I am sure you will agree with me our thanks are justly due.

The record of their labours and of the voluminous evidence which they obtained, to which evidence indeed several of us here contributed, is embodied in a 'blue-book' of more than usual interest and instruction. I commend the perusal of it not only to professed physicists, but to all scientific teachers and their advanced students, that they may learn what accuracy is, and what it is practically wanted for.

It is not for me as your President to touch at any length upon outside or political topics, but briefly I venture to think that the year 1898 has been a year of no ordinary good fortune, at least to the Anglo-Saxon race. A year, and a Government, which has witnessed and assisted not only the beginning of a National Physical Laboratory, but such great outside events as the Imperial expansion and cooperation of America, (for America, and not the United States, it will now have to be called), the liberation of Crete, and the restoration of the civilization of Upper Egypt, is a year upon which we can look back with satisfaction, and is a Government which, irrespective of party, we may legitimately congratulate, and even, though that I admit is unusual, thank.

Would that we could add to its laurels the commencement on wise lines of a real and comprehensive, dignified and progressive, genuine University of London.

PART II.—*On Opacity.*

MY attention has recently been called to the subject of the transmission of electromagnetic waves by conducting dielectrics—in other words, to the opacity of imperfectly conducting material to light. The question arose when an attempt was being made to signal inductively through a stratum of earth or sea, how far the intervening layers of moderately conducting material were able to act as a screen; the question also arises in the transmission of Hertz waves through partial conductors, and again in the transparency of gold-leaf and other homogeneous substances to light.

The earliest treatment of such subjects is due of course to Clerk Maxwell thirty-four years ago, when, with unexampled genius, he laid down the fundamental laws for the propagation of electric waves in simple dielectrics, in crystalline media, and in conducting media. He also realised there was some strong analogy between the transmission of such waves through space and the transmission of pulses of current along a telegraph-wire. But naturally at that early date not every detail of the investigation was equally satisfactory and complete.

Since that time, and using Maxwell as a basis, several mathematicians have developed the theory further, and no one with more comprehensive thoroughness than Mr. Oliver Heaviside, who has gone into these matters with extraordinarily clear and far vision. I may take the opportunity of calling or recalling to the notice of the Society, as well as of myself, some of the simpler developments of Mr. Heaviside's theory and manner of unifying phenomena and processes at first sight apparently different; but first I will deal with the better-known aspects of the subject.

Maxwell deals with the relation between conductivity and opacity in his Art. 798 and on practically to the end of that famous chapter xx. ('Electromagnetic Theory of Light'). He discriminates, though not very explicitly or obtrusively, between the two extreme cases, (1) when inductive capacity

or electric inductivity is the dominant feature of the medium—when, for instance, it is a slightly conducting dielectric, and (2) the other extreme case, when conductivity is the predominant feature.

The equation for the second case, that of predominant conductivity, is

$$\frac{d^2\mathrm{F}}{dx^2}=\frac{4\pi\mu}{\sigma}\frac{d\mathrm{F}}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

F being practically any vector representing the amplitude of the disturbance; for since we need not trouble ourselves with geometrical considerations such as the oblique incidence of waves on a boundary &c., we are at liberty to write the ∇ merely as d/dx, taking the beam parallel and the incidence normal.

No examples are given by Maxwell of the solution of this equation, because it is obviously analogous to the ordinary heat diffusion fully treated by Fourier.

Suffice it for us to say that, taking F at the origin as represented by a simple harmonic disturbance $\mathrm{F}_0=e^{ipt}$, the solution of equation (1)

$$\frac{d^2\mathrm{F}}{dx^2}=\frac{4\pi\mu ip}{\sigma}\mathrm{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1')$$

is
$$\mathrm{F}=\mathrm{F}_0\,e^{-\mathrm{Q}x}=e^{-\mathrm{Q}x+ipt},$$

where
$$\mathrm{Q}=\sqrt{\left(\frac{4\pi\mu ip}{\sigma}\right)}=\sqrt{\frac{2\pi\mu p}{\sigma}}\,.\,(1+i);$$

wherefore

$$\mathrm{F}=e^{-\left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}x}\cos\left(pt-\left(\frac{2\pi\mu p}{\sigma}\right)^{\frac{1}{2}}x\right), \quad . \quad . \quad . \quad (2)$$

an equation which exhibits no true elastic wave propagation at a definite velocity, but a trailing and distorted progress, with every harmonic constituent going at a different pace, and dying out at a different rate; in other words, the *diffusion* so well known in the case of the variable stage of heat-conduction through a slab.

In such conduction the gain of heat by any element whose heat capacity is $c\rho dx$ is proportional to the difference of the

temperature gradient at its fore and aft surfaces, so that

$$c\rho\, dx \frac{d\theta}{dt} = d \cdot k \frac{d\theta}{dx},$$

or, what is the same thing,

$$\frac{d^2\theta}{dx^2} = \frac{c\rho}{k} \frac{d\theta}{dt},$$

the same as the equation (1) above; wherefore the constant $c\rho/k$, the *reciprocal* of the thermometric conductivity, takes the place of $4\pi\mu/\sigma$, that is, of electric conductivity; otherwise the heat solution is the same as (2). The 4π has come in from an unfortunate convention, but it is remarkable that the conductivity term is inverted. The reason of the inversion of this constant is that, whereas the *substance* conveys the heat waves, and by its conductivity aids their advance, the *æther* conveys the electric waves, and the substance only screens and opposes, reflects, or dissipates them.

This is the case applied to sea-water and low frequency by Mr. Whitehead in a paper which he gave to this Society in June 1897, being prompted thereto by the difficulty which Mr. Evershed and the Post Office had found in some trials of induction signalling at the Goodwin Sands between a coil round a ship at the surface and another coil submerged at a depth of 10 or 12 fathoms. It was suspected that the conductivity of the water mopped up a considerable proportion of the induced currents, and Mr. Whitehead's calculation tended, or was held to tend, to support that conclusion.

To the discussion Mr. Heaviside communicated what was apparently, as reported, a brief statement; but I learn that in reality it was a carefully written note of three pages, of which recently he has been good enough to lend me a copy. In that note he calls attention to a theory of the whole subject which in 1887 he had worked out and printed in his collected 'Electrical Papers,' but which has very likely been overlooked. It seems to me a pity that a note by Mr. Heaviside should have been so abridged in the reported discussion as to be practically useless; and I am permitted to quote it here as an appendix (p. 413).

Meanwhile, taking the diffusion case as applicable to sea-water with moderately low acoustic frequency, we see that the

induction effect decreases geometrically with the thickness of the oceanic layer, and that the logarithmic decrement of the amplitude of the oscillation is $\sqrt{\left(\frac{2\pi\mu p}{\sigma}\right)}$, where σ is the specific resistance of sea-water and $p/2\pi$ is the frequency.

Mr. Evershed has measured σ and found it 2×10^{10} C.G.S., that is to say $2\times10^{10}\,\mu$ square centim. per second; so putting in this value and taking a frequency of 16 per second, the amplitude is reduced to $1/e$th of what its value would have been at the same distance in a perfect insulator, by a depth

$$\sqrt{\frac{\sigma}{2\pi\mu p}}=\sqrt{\left(\frac{2\times10^{10}\mu}{2\pi\mu\times2\pi\times16}\right)}=\sqrt{\frac{10^{10}}{320}}=\frac{10^5}{18}\text{ centim.}$$
$$=55\text{ metres.}$$

Four or five times this thickness of intervening sea would reduce the result at the 16 frequency to insignificance (each 55-metre-layer reducing the energy to $\frac{1}{7}$ of what entered it); but if the frequency were, say, 400 per second instead of 16 it would be five times more damped, and the damping thickness (the depth reducing the amplitude in the ratio $e:1$) would in that case be only eleven metres.

It is clear that in a sea 10 fathoms (or say 20 metres) deep the failure to inductively operate a "call" responding to a frequency of 16 per second was *not* due to the screening effect of sea-water *.

Maxwell, however, is more interested in the propagation of actual light, that is to say, in waves whose frequency is about 5×10^{14} per second; and for that he evidently does not consider that the simple diffusion theory is suitable. It certainly is not applicable to light passing through so feeble a conductor as salt water. He attends mainly therefore to the other and more interesting case, where electric inductive-capacity predominates over the damping effect of conductivity, and where true waves therefore advance with an

* I learn that the ship supporting the secondary cable was of metal, and that the primary or submerged cable was sheathed in uninsulated metal, viz. in iron, which would no doubt be practically short-circuited by the sea-water. Opacity of the medium is in that case a superfluous explanation of the failure, since a closed secondary existed close to both sending and receiving circuit.

approximately definite velocity

$$v = \frac{1}{\sqrt{\mu K}};$$

though it is to be noted that the slight sorting out of waves of different frequency, called dispersion, is an approximation to the case of pure diffusion where the speed is as the square root of the frequency, and is accompanied, moreover, as it ought to be, by a certain amount of differential or selective absorption.

To treat the case of waves in a conductor, the same damping term as before has to be added to the ordinary wave equation, and so we have

$$\frac{d^2F}{dx^2} = \mu K \frac{d^2F}{dt^2} + \frac{4\pi\mu}{\sigma}\frac{dF}{dt}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Taking $F_0 = e^{ipt}$ again, it may be written

$$\frac{d^2F}{dx^2} = \left(-\mu K p^2 + \frac{4\pi\mu i p}{\sigma}\right)F, \quad . \quad . \quad . \quad . \quad (3')$$

the same form as equation (1′); so the solution is again

$$F = e^{-Qx + ipt},$$

with Q^2 equal to the coefficient of F in (3′). Maxwell, however, does not happen to extract the square root of this quantity, but, assuming the answer to be of the form (for a simply harmonic disturbance) [modifying his letters, vol. ii. § 798]

$$e^{-rx}\cos(pt - qx),$$

he differentiates and equates coefficients, thus getting

$$q^2 - r^2 = \mu K p^2, \qquad 2rq = \frac{4\pi\mu p}{\sigma},$$

as the conditions enabling it to satisfy the differential equation. This of course gives for the logarithmic decrement, or coefficient of absorption,

$$r = \frac{2\pi\mu}{\sigma} \cdot \frac{p}{q},$$

p/q being precisely the velocity of propagation of the train of waves. Though not exactly equal to $1/\sqrt{\mu K}$, the true velocity of wave propagation, except as a first approximation, in an absorbing medium, yet practically this velocity p/q or λ/T

is independent of the frequency except in strongly absorbent substances where there are dispersional complications; and so the damping is, in simple cases, practically independent of the frequency too.

With this simple velocity in mind Maxwell proceeds to apply his theory numerically to gold-leaf, calculating its theoretical transparency, and finding, as every one knows, that it comes out discordant with experiment, being out of all comparison * smaller than what experiment gives.

But then it is somewhat surprising to find gold treated as a substance in which conductivity does not predominate over specific inductive capacity.

The differential equation is quite general and applies to any substance, and since the solution given is a true solution, it too must apply to any substance when properly interpreted; but writing it in the form just given does not suggest the full and complete solution. It seems to apply only to slightly damped waves, and indeed, Maxwell seems to consider it desirable to rewrite the original equation with omission of K, for the purpose of dealing with good conductors.

By a slip, however, he treats gold for the moment as if it belonged to the category of poor conductors, and as if absorption in a thickness such as gold-leaf could be treated as a moderate damping of otherwise progressive waves.

The slip was naturally due to a consideration of the extreme frequency of light vibrations; but attention to the more complete expression for the solution of the same differential equation, given in 1887 by Mr. Heaviside and quoted in the note to this Society above referred to, puts the matter in a proper position. Referring to his 'Electrical Papers,' vol. ii. p. 422, he writes down the general value of the coefficient of absorption as follows (translating into our notation)

$$r=\frac{p}{v\sqrt{2}}\left\{\left[1+\left(\frac{4\pi}{\sigma p\mathrm{K}}\right)^2\right]^{\frac{1}{2}}-1\right\}^{\frac{1}{2}}$$

without regard to whether the conductivity of the medium is large or small; where v is the undamped or true velocity of wave propagation in the medium $(\mu\mathrm{K})^{-\frac{1}{2}}$.

* The fraction representing the calculated transmission by a film half a wave thick has two thousand digits in its denominator: see below, top of p. 370.

Of course Maxwell could have got this expression in an instant by extracting the square root of the quantity Q, the coefficient of F in equation (3′) written above. I do not suppose that there is anything of the slightest interest from the mathematician's point of view, the interest lies in the physical application; but as this is not a mathematical Society it is permissible, and I believe proper, to indicate steps for the working out of the general solution of equation (3) by extracting the square root of the complex quantity Q.

The equation is

$$\frac{d^2F}{dx^2}+\left(\mu Kp^2-\frac{4\pi\mu ip}{\sigma}\right)F=0,$$

and the solution is

$$F=e^{-Qx+ipt},$$

where

$$Q=\sqrt{-\mu Kp^2+\frac{4\pi\mu ip}{\sigma}}=\alpha+i\beta \text{ say.}$$

Squaring we get, just as Maxwell did,

$$\alpha^2-\beta^2=-\mu Kp^2, \quad 2\alpha\beta=+\frac{4\pi\mu p}{\sigma}.$$

Squaring again and adding

$$(\alpha^2+\beta^2)^2=(\alpha^2-\beta^2)^2+4\alpha^2\beta^2=\mu^2K^2p^4+\frac{16\pi^2\mu^2p^2}{\sigma^2}$$

$$\therefore \quad \alpha^2+\beta^2=\mu Kp^2\left\{1+\left(\frac{4\pi}{Kp\sigma}\right)^2\right\}^{\frac{1}{2}},$$

wherefore

$$2\beta^2=\mu Kp^2\left\{\sqrt{\left(1+\left(\frac{4\pi}{Kp\sigma}\right)^2\right)}+1\right\}, \quad . \quad . \quad (4)$$

and $2\alpha^2$ = the same with the last sign negative,

or $$\alpha=p\sqrt{(\tfrac{1}{2}\mu K)}\Big\}\left[1+\left(\frac{4\pi}{\sigma pK}\right)^2\Big\}^{\frac{1}{2}}-1\right]^{\frac{1}{2}}, \quad . \quad . \quad . \quad (5)$$

which is the logarithmic decrement of the oscillation per unit of distance, or the reciprocal of the thickness which reduces the amplitude in the ratio $1:e$ (or the energy to $\frac{1}{e}$) of the value it would have at the same place without damping.

Using these values for α and β, the radiation-vector in general, after passing through any thickness x of any medium

whose magnetic permeability and other properties are *constant*, is

$$F = F_0 e^{-\alpha x} \cos(pt - \beta x), \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the speed of advance of the wave-train being p/β.

Now not only the numerical value but the form of this damping constant α depends on the magnitude of the numerical quantity $\frac{4\pi}{\sigma p K}$, which may be called the critical number*, and may also be written

$$\frac{4\pi\mu_0 v_0^2}{p\sigma K/K_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where K, the absolute specific inductive capacity of the medium, is replaced by its relative value in terms of K_0 for vacuum, and by $\frac{1}{\sqrt{K_0\mu_0}}$ = the velocity of light *in vacuo* $= v_0$.

Now for all ordinary frequencies and good conductors this critical number is large; and in that case it will be found that

$$\alpha = p\sqrt{\frac{\mu K}{2}} \cdot \sqrt{\frac{4\pi}{\sigma p K}} = \sqrt{\frac{2\pi\mu p}{\sigma}},$$

and that β is identically the same. This represents the simple diffusion case, and leads to equation (2).

On the other hand, for luminous frequency and bad conductors, the critical quantity is small, and in that case

$$\alpha = p\sqrt{(\tfrac{1}{2}\mu K)}\left\{1 + \tfrac{1}{2}\left(\frac{4\pi}{\sigma p K}\right)^2 - 1\right\}^{\frac{1}{2}}$$

$$= \tfrac{1}{2}p\sqrt{(\mu K)} \cdot \frac{4\pi}{\sigma p K} = \frac{2\pi\mu v}{\sigma},$$

while

$$\beta = p\sqrt{\mu K} = \frac{p}{v},$$

giving the solution

$$F = F_0 e^{-\frac{2\pi\mu v}{\sigma}x} \cos p\left(t - \frac{x}{v}\right). \quad . \quad . \quad (8)$$

* An instructive mode of writing α and β in general is given in (11″) or (12″) below, where the above critical number is called $\tan\epsilon$:—

$$\alpha v\sqrt{\cos\epsilon} = p\sin\tfrac{1}{2}\epsilon,$$
$$\beta v\sqrt{\cos\epsilon} = p\cos\tfrac{1}{2}\epsilon.$$

This expresses the transmission of light through imperfect insulators, and is the case specially applied by Maxwell to calculations of opacity. Its form serves likewise for telegraphic signals or Hertz waves transmitted by a highly-conducting aerial wire ; the damping, if any, is independent of frequency and there is true undistorted wave-propagation at velocity $v=1/\sqrt{(LS)}$; the constants belonging to unit length of the wire. The current (or potential) at any time and place is

$$C=C_0 e^{-\frac{Rx}{2Lv}} \cos p(t-x\sqrt{LS}). \quad . \quad . \quad . \quad (9)$$

The other extreme case, that of diffusion, represented by equation (2), is analogous to the well-known transmission of slow signals by Atlantic cables, that is by long cables where resistance and capacity are predominant, giving the so-called KR law (only that I will write it RS),

$$C = C_0 e^{-\sqrt{(\frac{1}{2}pRS)}x} \cos \{pt-\sqrt{(\tfrac{1}{2}pRS)}x\}; \quad . \quad . \quad (10)$$

wherefore the damping distance in a cable is

$$x_0 = \sqrt{\left(\frac{2}{pRS}\right)}.$$

Thus, in comparing the cable case with the penetration of waves into a conductor and with the case of thermal conduction, the following quantities correspond :

$$\tfrac{1}{2}pRS, \qquad \frac{4\pi\mu p}{2\sigma}, \qquad \frac{pc\rho}{2k}.$$

$c\rho$ is the heat-capacity per unit volume, S is the electric capacity per unit length ; k is the thermal conductivity per unit volume, $1/R$ is the electric conductance per unit length. So these agree exactly ; but in the middle case, that of waves entering a conductor, there is a notable inversion, representing a real physical fact. $4\pi\mu$ may be called the density and may be compared with ρ or with 1 S, that is with elasticity $\div v^2$; but σ is the *resistance* per unit volume instead of the conductance. The reason of course is that whereas good conductivity helps the cable-signals or the heat along, it by no means helps the waves into the conductor. Conductivity aids their slipping *along* the boundary of a conductor, but it retards their passing *across* the boundary and entering a

conductor. As regards waves entering a conductor, the effect of conductivity is a screening effect, not a transmitting effect, and it is the bad conductor which alone has a chance of being a transparent medium.

It may be convenient to telegraphists, accustomed to think in terms of the "KR-law" and comparing equations (2) and (10), to note that the quantity $4\pi\mu\,\sigma$—that is, practically, the specific conductivity in electromagnetic measure (multiplied by a meaningless 4π because of an unfortunate initial convention)—takes the place of KR (*i. e.* of RS), but that otherwise the damping-out of the waves as they enter a good conductor is exactly like the damping-out of the signals as they progress through a cable; or again as electrification travels along a cotton thread, or as a temperature pulse makes its way through a slab; and yet another case, though it is different in many respects, yet has some similarities, viz. the ultimate distance the melting-point of wax travels along a bar in Ingenhousz's conductivity apparatus,—the same law of inverse square of distance for effective reach of signal holding in each case.

Now it is pointed out by Mr. Heaviside in several places in his writings that, whereas the transmission of high-frequency waves by a nearly transparent substance corresponds by analogy to the conveyance of Hertz waves along aerial wires (or along cables for that matter, if sufficiently conducting), and whereas the absorption of low-frequency waves by a conducting substance corresponds, also by analogy, to the diffusion of pulses along a telegraph-cable whose self-induction is neglected—its resistance and capacity being prominent,—the intermediate case of waves of moderate frequency in a conductor of intermediate opacity corresponds to the more general cable case where self-induction becomes important and where leakage also must be taken into account; because it is leakage conductance that is the conductance of the dielectric concerned in plane waves. This last is therefore a real, and not only an analogic, correspondence.

Writing $R_1\ S_1\ L_1\ Q_1$ for the resistance, the capacity ("permittance"), the inductance, and the leakage-conductance ("leakance") respectively, per unit length, the general equations to cable-signalling are given in Mr. Heaviside's

'Electromagnetic Theory' thus :—

$$S_1\frac{dV}{dt}+Q_1V+\frac{dC}{dx}=0,\quad L_1\frac{dC}{dt}+R_1C+\frac{dV}{dx}=0\,;$$

or for a simple harmonic disturbance,

$$\frac{d^2V}{dx^2}=(R_1+ipL_1)(Q_1+ipS_1)V \quad . \quad . \quad . \quad (11)$$
$$=(\alpha+i\beta)^2V,$$

whose solution therefore is

$$V=V_0\,e^{-\alpha x}\cos(pt-\beta x)\,*. \quad . \quad . \quad . \quad (11')$$

There are several interesting special cases:—

The old cable theory of Lord Kelvin is obtained by omitting both Q and L; thus getting equation (2).

The transmission of Hertz waves along a perfectly-conducting insulated wire is obtained by omitting Q and R; the speed of such transmission being $1/\sqrt{(L_1S_1)}$. Resistance in the wire brings it to the form (9), where the damping depends on the ratio of the capacity constant RS to the self-induction constant L/S; because the index $R/2Lv$ equals half the square root of this ratio; but it must be remembered that R has the throttled value due to merely superficial penetration. The case is approximated to in telephony sometimes.

A remarkable case of undistorted (though attenuated) transmission through a cable (discovered by Mr. Heaviside, but not yet practically applied) is obtained by taking

$$R/L=Q/S=r\,;$$

the solution being then

$$V=e^{-\frac{rx}{v}}f\left(t-\frac{x}{v}\right),$$

due to $f(t)$ at $x=0$. All frequencies are thus treated alike,

* I don't know whether the following simple general expression for α and β has been recorded by anyone: writing $R/pL=\tan\epsilon$ and $Q/pS=\tan\epsilon'$,

$$\alpha \text{ or } \beta=\frac{p}{v}\cdot\frac{\sin \textit{or} \cos\frac{1}{2}(\epsilon+\epsilon')}{(\cos\epsilon\cos\epsilon')^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11'')$$

which is shorter than (12).

and a true velocity of transmission makes its reappearance. This is what he calls his "distortionless circuit," which may yet play an important part in practice.

And lastly, the two cases which for brevity may be treated together, the case of perfect insulation, $Q=0$, on the one hand, and the case of perfect wire conduction, $R=0$, on the other. For either of these cases the general expression

$$\alpha^2 \text{ or } \beta^2 = \tfrac{1}{2}p^2 L_1 S_1 \left[\left\{1+\left(\frac{R}{pL}\right)^2\right\}^{\frac{1}{2}} \left\{1+\left(\frac{Q}{pS}\right)^2\right\}^{\frac{1}{2}} \pm \left\{\frac{RQ}{p^2 LS} - 1\right\} \right] \quad (1$$

becomes exactly of the form (4) or (5) reckoned above for the general screening-effect, or opacity, of conducting media in space.

For the number which takes the place of the quantity there called the critical number, namely either R/pL or Q/pS, the other being zero, we may write $\tan\epsilon$; in which case the above is

$$\alpha^2 \text{ or } \beta^2 = \tfrac{1}{2}p^2 L_1 S_1 (\sec\epsilon \mp 1)\,; \quad . \quad . \quad . \quad . \quad . \quad (12')$$

or, rewriting in a sufficiently obvious manner, with $2\pi/\lambda$ for p/v if we choose,

$$\alpha = \frac{p \sin\frac{1}{2}\epsilon}{v(\cos\epsilon)^{\frac{1}{2}}}, \qquad \beta = \frac{p\cos\frac{1}{2}\epsilon}{v(\cos\epsilon)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad (12'')$$

Instead of attending to special cases, if we attend to the general cable equation (11) as it stands, we see that it is more general than the corresponding equation (3) to waves in space, because it contains the extra possibility R of wire resistance, which does not exist in free space.

Mr. Heaviside, however, prefers to unify the whole by the introduction of a hypothetical and as yet undiscovered dissipation-possibility in space, or in material bodies occupying space, which he calls magnetic conductance, and which, though supposed to be non-existent, may perhaps conceivably represent the reciprocal of some kind of hysteresis, either the electric or the magnetic variety. Calling this g, (gH^2 is to be the dissipation term corresponding with RC^2), the equation to waves in space becomes

$$\nabla^2 F = (g + ip\mu)\left(\frac{4\pi}{\sigma} + ipK\right) F, \quad . \quad . \quad . \quad (13)$$

just like the general cable case. And a curious kind of transparency, attenuation without distortion, would belong to a medium in which both conductivities coexisted in such proportion that $g : \mu = 4\pi k : \mathrm{K}$; for g would destroy H just as k destroys E.

In the cable, F may be either current or potential, and $\mathrm{LS}v^2 = 1$. In space, F may be either electric or magnetic intensity, and $\mu \mathrm{K} v^2 = 1$; but observe that g takes the place not of Q but of R, while it is $4\pi/\sigma$ that takes the place of Q. Resistance in the wire and electric conductivity in space do not produce similar effects. If there is any analogue in space to wire resistance it is magnetic not electric conductivity.

The important thing is of course that the wire does not convey the energy but dissipates it, so that the dissipation by wire-resistance and the dissipation by space-hysteresis to that extent correspond. The screening effect of space-conductivity involves the very same dielectric property as that which causes leakage or imperfect insulation of the cable core.

Returning to the imaginary magnetic conductivity, let us trace what its effects would be if it existed, and try to grasp it. It effect would be to kill out the magnetism of permanent magnets in time, and generally to waste away the energy of a static magnetic field, just as resistance in wires wastes the energy of an unmaintained current and so kills out the magnetism of *its* field. I spoke above as if it were conceivable that such magnetic conductivity could actually in some degree exist, likening it to a kind of hysteresis ; but hysteresis—the enclosure of a loop between a to and fro path —is a phenomenon essentially associated with *fluctuations*, and cannot exist in a steady field with everything stationary. Admitted : but then the molecules are not stationary, and the behaviour of molecules in the Zeeman and Righi phenomena, or still more strikingly in the gratuitous radiations discovered by Edmond Becquerel, and more widely recognized by others, especially by Monsieur et Madame Curie, (not really gratuitous but effected probably by conversion into high-pitched radiation of energy supplied from low-pitched sources),—the way molecules of absorbent substances behave, seems to render

possible, or at least conceivable, something like a minute magnetic conductivity in radiative or absorptive substances. Mr. Heaviside, however, never introduced it as a physical fact for which there was any experimental evidence, but as a physical possibility and especially as a mathematical auxiliary and unifier of treatment, and that is all that we need here consider it to be; but we may trace in rather more detail its effect if it did exist.

Suppose the magnetism of a magnet decayed, what would happen to its lines of force? They would gradually shrink into smaller loops and ultimately into molecular ones. The generation of a magnetic field is always the opening out of previously existing molecular magnetic loops; there is no such thing as the creation of a magnetic field, except in the sense of moving it into a fresh place or expanding it over a wider region *. So also the destruction of a magnetic field merely means the shrinkage of its lines of force (or lines of induction, I am not here discriminating between them). Now consider an electric current in a wire:—a cylindrical magnetic field surrounds it, and if the current gradually decreases in strength the magnetic energy gradually sinks into the wire as its lines slowly collapse. But observe that the *electric* energy of the field remains unchanged by this process: if the wire were electrostatically charged it would remain charged, its average potential can remain constant. Let the wire for instance be perfectly conducting, then the current needs no maintenance, the potential might be uniform (though in general there would be waves running to and fro), and both the electric and magnetic fields continue for ever, unless there is some dissipative property in space.

Two kinds of dissipative property may be imagined in matter filling space: first, and most ordinary, an electric conductivity or simple leakage, the result of which will be to equalize the potential throughout space and destroy the electric field, without necessarily affecting the magnetic field, and so without stopping the steady circulation of the current manifested by that field. The other dissipative property in space that could be imagined would be magnetic conductivity; the result of which would be to shrink all the circular lines of

* This may be disagreed with.

magnetic force slowly upon the wire, thus destroying the magnetic field, and with it (by the circuital relation) the current; but leaving the electrostatic potential and the electric field unchanged. And this imaginary effect of the medium in surrounding space is exactly the real effect caused by what is called electric resistance in the wire *.

Now for a simply progressive undistorted wave, *i. e.* one with no character of diffusion about it, but all frequencies travelling at the same quite definite speed $1/\sqrt{\mu K}$, it is essential that the electric and magnetic energies shall be equal. If both are weakened *in the same proportion*, the wave-energy is diminished, and the pulse is said to be "attenuated," but it continues otherwise uninjured and arrives "undistorted," that is, with all its features intact and at the same speed as before, but on a reduced scale in point of size.

This is the case of Mr. Heaviside's "distortionless circuit" spoken of above, and its practical realization in cables, though it would not at once mean Atlantic telephony, would mean greatly improved signalling, and probably telephony through shorter cables. In a cable the length of the Atlantic the attenuation would be excessive, unless the absence of distortion were secured by increasing rather the wire-conductance than the dielectric leakage; but, unless excessive, simple attenuation does no serious harm. Articulation depends on the features of the wave, and the preservation of the features demands, by Fourier's analysis, the transmission of every frequency at the same rate.

But now suppose any cause diminishes one of the two fields without diminishing the other: for instance, let the electric field be weakened by leakage alone, or let the magnetic field be weakened by wire-resistance alone, then what happens? The preservation of E and the diminution of H, to take the latter—the ordinary—case, may be regarded as a superposition on the advancing wave of a gradually growing reverse field of intensity δH; and, by the relation $E=\mu v H$,

* There is this difference, that in the real case the heat of dissipation appears locally in the wire, whereas in the imaginary case it appears throughout the magnetically conducting medium; but I apprehend that in the imaginary case the lines would still shrink, by reason of molecular loops being pinched off them.

this reversed field, for whatever it is worth, must mean a gradually growing wave travelling in the reverse direction.

The ordinary wave is now no longer left alone and uninjured, it has superposed upon itself a more or less strong reflected wave, a reflected wave which constantly increases in intensity as the distance along the cable, or the penetration of the wave into a conducting medium, increases; all the elementary reflected waves get mixed up by re-reflexion in the rear, constituting what Mr. Heaviside calls a diffusive "tail"; and this accumulation of reflected waves it is which constitutes what is known as "distortion" in cables, and what is known as "opacity" inside conducting dielectrics.

There is another kind of opacity, a kind due to heterogeneousness, not connected with conductivity but due merely to a change in the constants K and μ,—properly a kind of translucency, a scattering but not a dissipation of energy,—like the opacity of foam or ground glass.

This kind of opacity is an affair of boundaries and not of the medium itself, but after all, as we now see, it has features by no means altogether dissimilar to the truer kind of opacity. Conducting opacity is due to reflexion, translucent opacity is due to reflexion,—to irregular reflexion as it is called, but of course there is nothing irregular about the reflexion, it is only the distribution of boundaries which is complicated, the reflexion is as simple as ever;—except, indeed, to some extent when the *size* of the scattering particles has to be taken into account and the blue of the sky emerges. But my point is that this kind of opacity also is after all of the reflexion kind, and the gradual destruction of the advancing wave—whether it be by dust in the air or, as Lord Rayleigh now suggests, perhaps by the discrete molecules themselves, by the same molecular property as causes refraction and dispersion—must result in a minute distortion and a mode of wave propagation not wholly different from cable-signalling or from the transmission of light through conductors. So that the red of the sunset sky and the green of gold-leaf may not be after all very different; nor is the arrival-curve of a telegraph signal a wholly distinct phenomenon.

There is a third kind of opacity, that of lampblack, where the molecules appear to take up the energy direct, converting

it into their own motion, that is into heat, and where there appears to be little or nothing of the nature of reflexion. I am not prepared to discuss that kind at present.

It is interesting to note that in the most resisting and capacious cable that ever was made, where all the features of every wave arrive as obliterated as if one were trying to signal by heat-pulses through a slab, that even there the *head* of every wave travels undistorted, with the velocity of light, and suffers nothing but attenuation; for the superposed reversed field is only called out by the arrival of the direct pulse, and never absolutely reaches the strength of the direct field. The attenuation may be excessive, but the signal is there in its right time if only we have a sensitive enough instrument to detect it; though it would be practically useless as a signal in so extreme a case, being practically *all* tail.

Nothing at all reaches the distant end till the light-speed-time has elapsed; and the light-speed-time in a cable depends on the μ and K of its insulating sheath, depends, if that is not simply cylindrical, on the product of its self-inductance and capacity per unit length; but at the expiration of the light-speed-time the head of the signalling pulse arrives, and neither wire-resistance nor insulation-leakage, no, nor magnetic-conductivity, can do anything either to retard it or to injure its sharpness: they can only enfeeble its strength, but they can do that very effectually.

The transmitter of the pulse is self-induction in conjunction with capacity: the chief practical enfeebler of the pulse is wire-resistance in conjunction with capacity; and before Atlantic telephony is possible (unless a really distortionless cable is forthcoming) the copper core of an ordinary cable will have to be made much larger. Nothing more is wanted in order that telephony to America may be achieved. There may be practical difficulties connected with the mechanical stiffness of a stout core and the worrying of its guttapercha sheath, and these difficulties may have to be lessened by aiming at distortionless conditions—it is well known also that for high frequencies a stout core must be composed of insulated strands unless it is hollow—but when such telephony is accomplished, I hope it will be recollected that the full and complete principles of it and of a great deal else connected with tele-

graphy have been elaborately and thoroughly laid down by Mr. Heaviside.

There is a paragraph in Maxwell, concerning the way a current rises in a conductor and affects the surrounding space, which is by no means satisfactory: it is Art. 804. He takes the current as starting all along the wire, setting up a sheath of opposition induced currents in the surrounding imperfectly insulating dielectric, which gradually diffuse outwards and die away, leaving at last the full inductive effect of the core-current to be felt at a distance. Thus there is supposed to be a diffusion of energy outwards from the wire, which he likens to the diffusion of heat.

But, as Mr. Heaviside has shown, the true phenomenon is the transmission of a wave in the space surrounding the wire—a plane wave if the wire is perfectly conducting, a slightly coned wave if it resists,—a wave-front perpendicular to the wire and travelling along it,—*a sort of beam of dark light with the wire as its core.*

Telegraphic signalling and optical signalling are similar; but whereas the beam of the heliograph is abandoned to space and must go straight except for reflexion and refraction, the telegraphic beam can follow the sinuosities of the wire and be guided to its destination.

If the medium conducts slightly it will be dissipated *in situ*; but if the wire conducts imperfectly, a minute trickle of energy is constantly directed inwards radially towards the wire core, there to be dissipated as heat. Parallel to the wire flows the main energy stream, but there is a small amount of tangential grazing and inward flow. The initial phenomenon does not occur in the wire, gradually to spread outwards, but it occurs in the surrounding medium, and a fraction of it gradually converges inwards. The advancing waves are not cylindrical but plane waves, and though the diffusing waves are cylindrical they advance inwards, not outwards.

I will quote from a letter of Mr. Heaviside's:—"The easiest way to make people understand is, perhaps, to start with a conducting dielectric with plane waves in it without wires, [thus getting] one kind of attenuation and distortion. Then introduce wires of *no* resistance; there is no difference except

in the way the lines of force distribute [enabling the wires to guide the plane waves]. Then introduce magnetic conductivity in the medium, [thereby getting] the other kind of attenuation and distortion. Transfer it to the wires, making it electrical resistance. Then abolish the first electric conductivity, and you have the usual electric telegraph."

Opacity of Gold-leaf.

Now returning to the general solution (5) let us apply it to calculate the opacity of gold-leaf to light.

Take $\sigma = 2000\,\mu$ square centim. per sec.,
$p = 2\pi \times 5 \times 10^{14}$ per sec.;

then the critical quantity $4\pi/p\sigma\mathrm{K}$ or (7) is

$$\frac{2 \times 9 \times 10^{20}}{5 \times 10^{14} \times 2000\,\mathrm{K}/\mathrm{K}_0} = \frac{1800}{\mathrm{K}/\mathrm{K}_0}.$$

This number is probably considerably bigger than unity (unless, indeed, the specific inductive capacity K/K_0 of gold is immensely large, which may indeed be the case—refractive index 40, for instance,—only it becomes rather difficult to define); so that, approximately,

$$\alpha = \sqrt{\left(\frac{2\pi\mu p}{\sigma}\right)} = \sqrt{\frac{40 \times 5 \times 10^{14}}{2000}} = \sqrt{10^{13}} = 3 \times 10^{6};$$

or the damping distance is

$$\tfrac{1}{30} \times 10^{-5} \text{ centim.} = \tfrac{1}{3} \text{ microcentimetre},$$

whereas the wave-length in air is

$$6 \times 10^{-5} \text{ centim.} = 60 \text{ microcentimetres.}$$

The damping distance is therefore getting nearer to the right order of magnitude, but the opacity is still excessive.

A common thickness for gold-leaf is stated to be half a wave-length of light; that is to say, 90 times the damping distance. Hence the amplitude of the light which gets through a half-wave thickness of gold is e^{-90} of that which enters; and that is sheer opacity.

[Maxwell's calculation in Art. 798, carried out numerically, makes the damping

$$e^{-\frac{2\pi\mu v}{\sigma}x}, \;=\; \exp.\,(-10^8 x) \text{ for gold,}$$

see equation (8) above; or, for a thickness of half a wave-length, 10^{-1000}, which is billions of billions of billions (indeed a number with 960 digits) times greater opacity than what we have here calculated, and is certainly wrong.]

It must, however, be granted, I think, that the green light that emerges from gold-leaf is not properly transmitted; it is light re-emitted by the gold *. The incident light, say the red, is all stopped by a thickness less than half a wave-length. The green light may conceivably be due to atoms vibrating fairly in concordance, and not calling out the conducting opacity of the metal. If the calculated opacity, notwithstanding this, is still too great, it is no use assuming a higher conductivity at higher frequency, for that would act the wrong way. What must be assumed is either some special molecular dispersion theory, or else greater specific resistance for oscillations of the frequency which get through; nor must the imaginative suggestion made immediately below equation (13) be altogether lost sight of.

There is, however, the possibility mentioned above that the relative specific inductive capacity of gold, $K\,K_0$, if a meaning can be attached to it, may be very large, perhaps (though very improbably, see Drude, Wied. *Ann.* vol. xxxix. p. 481) comparable with 1800. Suppose for a moment that it is equal to 1800; then the value of the critical quantity (7) is 1 and the value of α is

$$p\sqrt{\tfrac{1}{2}\mu K\times\cdot 4}=\frac{p}{v}\sqrt{\left(\frac{K}{5K_0}\right)}=\frac{2\pi\times 5\times 10^{14}}{3\times 10^{10}}\sqrt{360}$$
$$=19\times 10^{5},$$

which reduces the calculated opacity considerably, though still not enough.

In general, calling $K/K_0=c$, and writing the critical number $\dfrac{4\pi\mu v^2}{\sigma p c}$ as h/c, we have

$$\frac{2v^2}{p^2}\alpha^2=\tfrac{1}{2}\alpha^2/\beta^2=\lambda^2\alpha^2/2\pi^2=\sqrt{(h^2+c^2)}-c\,;$$

* This would be fluorescence, of course; and Dr. Larmor argues in favour of a simple ordinary exponential coefficient of absorption even in metals. See Phil. Trans. 1894, p. 738, § 27.

so that $a\lambda/2\pi$ ranges from $\sqrt{\tfrac{1}{2}h}$ when h/c is big, to $\tfrac{1}{2}h/\sqrt{c}$ when h/c is small.

Writing the critical number h/c as $\tan\epsilon$, the general value of a is given by

$$a\lambda = \pi\sqrt{2c(\sec\epsilon - 1)}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This is the ratio of the wave-length in air to the damping distance in the material in general; meaning by "the damping distance" the thickness which reduces the amplitude in the ratio $e:1$. (14) represents expression (5); compare with (12′).

Theory of a Film.

So far nothing has been said about the limitation of the medium in space, or the effect of a boundary, but quite recently Mr. Heaviside has called my attention to a special theory, a sort of Fresnel-like theory, which he has given for infinitely thin films of finite conductance; it is of remarkable simplicity, and may give results more in accordance with experiment than the theory of the universal opaque medium without boundary, hitherto treated: a medium in which really the source is immersed.

Let a film, not so thick as gold-leaf, but as thin as the black spot of a soap-bubble, be interposed perpendicularly between source and receiver. I will quote from 'Electrical Papers,' vol. ii. p. 385:—"Let a plane wave $E_1 = \mu v H_1$ moving in a nonconducting dielectric strike flush an exceedingly thin sheet of metal [so thin as to escape the need for attending to internal reflexions, or the double boundary, or the behaviour inside]; let $E_2 = \mu v H_2$ be the transmitted wave out in the dielectric on the other side, and $E_3 = -\mu v H_3$ be the reflected wave *.

* *General Principles.*—It may be convenient to explain here the principles on which Mr. Heaviside arrives at his remarkably neat expression for a wave-front in an insulating medium,

$$E = \mu v H,$$

or as it may be more fully and vectorially written,

$$V(\nu\mathbf{E}) = \mu v \mathbf{H},$$

where $\mathbf{E}$ is a vector representing the electric intensity (proportional to the electric displacement), $\mathbf{H}$ is the magnetic intensity, and ν is unit normal to the wave-front. $\mathbf{E}$ and $\mathbf{H}$ are perpendicular vectors in the

"At the sheet we have

$$E_1 + E_3 = E_2$$
$$H_1 + H_3 = H_2 + 4\pi kz E_2,$$

k being the conductivity of the sheet of thickness z. Therefore

$$\frac{E_2}{E_1} = \frac{H_2}{H_1} = \frac{E_1 + E_3}{E_1} = \frac{1}{1 + 2\pi\mu kzv}, \quad . \quad . \quad (15)$$

---

same plane, *i.e.* in the same phase, and **E H** ν are all at right angles to each other.

The general electromagnetic equations in an insulating medium are perhaps sufficiently well known to be, on Mr. Heaviside's system,

$$\text{curl}\,\mathbf{H} = K\dot{\mathbf{E}} \quad \text{and} \quad -\text{curl}\,\mathbf{E} = \mu\dot{\mathbf{H}},$$

where "curl" is the vector part of the operator ∇, and where Maxwell's vector-potential and other complexities have been dispensed with.

[In case these equations are not familiar to students I interpolate a parenthetical explanation which may be utilised or skipped at pleasure.

The orthodox definition of Maxwell's name "curl" is that **b** is called the curl of **a** when the surface-integral of **b** through an area is equal to the line-integral of **a** round its boundary, **a** being a vector or a component of a vector agreeing everywhere with the boundary in direction, and **b** being a vector or component of vector everywhere normal to the area. Thus it is an operator appropriate to a pair of looped or interlocked circuits, such as the electric and the magnetic circuits always are. The first of the above fundamental equations represents the fact of electromagnetism, specially as caused by displacement currents in an insulator, the second represents the fact of magneto-electricity, Faraday's magneto-electric induction, in any medium. Taking the second first, it states the fundamental law that the induced EMF in a boundary equals the rate of change in the lines of force passing through it; since the EMF or step of potential all round a contour is the line-integral of the electric intensity E round it, so that

$$\text{EMF} = \int_{\text{cycle}} E\,ds = -\frac{dN}{dt} = -\iint \dot{B}\,dS = -\iint \mu\dot{H}\,dS\,;$$

wherefore $-\mu\dot{H}$ equals the curl of E. (The statement of this second circuital law is entirely due to Mr. Heaviside; it is now largely adopted and greatly simplifies Maxwell's treatment, abolishing the need for vector potential.) [See, however, Appendix III. p. 386. *May* 1899.]

The first of the above two fundamental equations, on the other hand, depends on the fact that a current round a contour excites lines of magnetic force through the area bounded by it, and states the law that the total magnetomotive force, or line-integral of the magnetic intensity round the boundary, is equal to 4π times the total current through it; the total current being the "ampere-turns" of the practical Engineer.

H is reflected positively and E negatively. A perfectly conducting barrier is a perfect reflector; it doubles the magnetic force and destroys the electric force on the side containing the incident wave, and transmits nothing."

[I must here interpolate a remark to the effect that though it can hardly be doubted that the above boundary conditions (tangential continuity of both E and H) are correct, yet in general we cannot avoid some form of æther-theory

---

Expressing this law in terms of current density c, we write

$$\text{MMF} = \int_{\text{cycle}} \text{H}ds = 4\pi\text{C} = \iint 4\pi c d\text{S};$$

so always current-density represents the curl of the magnetic field due to it, or curl $\text{H}=4\pi c$.

Now in a conductor $c=k\text{E}$, but in an insulator $c=\dot{\text{D}}$, the rate of change of displacement or Maxwell's "displacement-current"; and the displacement itself is proportional to the intensity of the electric field, $\text{D}=\frac{\text{K}}{4\pi}\text{E}$; hence the value of current density in general is

$$c=k\text{E}+\frac{\text{K}}{4\pi}\dot{\text{E}},$$

whence in general

$$\text{curl H}=4\pi k\text{E}+\text{K}\dot{\text{E}}=(4\pi k+\text{K}p)\text{E},$$

and in an insulator the conductivity k is nothing.

The connexion between "curl" so defined and $\text{V}\nabla$ is explained as follows. The operator ∇ applied to a vector R whose components are X Y Z gives

$$\left(i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz}\right)(i\text{X}+j\text{Y}+k\text{Z}),$$

which, worked out, yields two parts

$$\text{S}\nabla\text{R} \text{ or } -\left(\frac{d\text{X}}{dx}+\frac{d\text{Y}}{dy}+\frac{d\text{Z}}{dz}\right),$$

also called convergence, and

$$\text{V}\nabla\text{R} \text{ or } i\left(\frac{d\text{Z}}{dy}-\frac{d\text{Y}}{dz}\right)+j\left(\frac{d\text{X}}{dz}-\frac{d\text{Z}}{dx}\right)+k\left(\frac{d\text{Y}}{dx}-\frac{d\text{X}}{dy}\right)$$

or say $i\xi+j\eta+k\zeta$, where $\xi\ \eta\ \zeta$ are the components of a spin-like vector ω. Now a theorem of Sir George Stokes shows that the normal component of ω integrated over any area is equal to the tangential component of R integrated all round its boundary; hence $\text{V}\nabla$ and curl are the same thing.

Whenever ω is zero it follows that R has no circulation but is the derivative of an ordinary single-valued potential function, whose $d\text{V}=\text{X}dx+\text{Y}dy+\text{Z}dz$. In electromagnetism this condition is by no means

when we have to lay down continuity conditions, and, according to the particular kind of æther-theory adopted so will the boundary conditions differ. My present object is to awaken a more general interest in the subject and to represent Mr. Heaviside's treatment of a simple case; but it must be understood that the continuity conditions appropriate to oblique incidence have been treated by other great mathematical physicists, notably by Drude, J. J. Thomson, and

---

satisfied. $\dot{E}$ and H or $\dot{H}$ and E are both full of circulation, and their circuits are interlaced. Fluctuation in E by giving rise to current causes H; fluctuation in H causes induced E.]

Now differentiating only in a direction normal to a plane wave advancing along x, the operator V∇ becomes simply id/dx when applied to any vector *in* the wave-front, the scalar part of ∇ being nothing.

So the second of the above fundamental equations can be written

$$-i\frac{dE}{dx}=\mu\frac{dH}{dt},$$

or

$$\frac{dE}{dH}=\mu iv;$$

so, ignoring any superposed constant fields of no radiation interest, E and H are vectors in the same phase at right angles to each other, and their tensors are given by $E=\mu vH$.

Similarly of course the other equation furnishes $H=KvE$; thus giving the ordinary $K\mu v^2=1$, and likewise the fact that the electric and magnetic energies per unit volume are equal, $\frac{1}{2}KE^2=\frac{1}{2}\mu H^2$.

A wave travelling in the opposite direction will be indicated by $E=-\mu vH$; hence, as is well known, if either the electric or the magnetic disturbance is reversed in sign the direction of advance is reversed too.

(The readiest way to justify the equation $E=\mu vH$, *à posteriori*, is to *assume* the two well-known facts obtained above, viz. that the electric and magnetic energies are equal in a true advancing wave, and that $v=1/\sqrt{\mu K}$; then it follows at once.)

Treatment of an insulating boundary.—At the boundary of a different medium without conductivity the tangential continuity of E and of H across the boundary gives us the equations

$$E_1+E_3=E_2$$
$$H_1+H_3=H_2,$$

where the suffix 1 refers to incident, the suffix 2 to transmitted, and the suffix 3 to reflected waves.

H_1+H_3 may be replaced by $\mu v(E_1-E_3)$, since the reflected wave is

Larmor, also by Lord Rayleigh, and it would greatly enlarge the scope of this Address if I were to try to discusss the difficult and sometimes controversial questions which arise. I must be content to refer readers interested to the writings of the Physicists quoted—especially I may refer to J. J. Thomson's 'Recent Researches,' Arts. 352 to 409, and to Larmor, Phil. Trans, 1895, vol. 186, Art. 30, and other places.]

Now apply this to an example. Take k for gold, as we have done before, to be $1/2000\,\mu$ seconds per square centim.

---

reversed; so we shall have, for the second of the continuity equations,

$$E_1 - E_3 = \frac{\mu_2 v_2}{\mu_1 v_1} E_2 = nm E_2;$$

n being the index of refraction, and m the relative inductivity. Hence, adding and subtracting,

$$\frac{E_2}{E_1} = \frac{2}{1+nm},$$

and

$$\frac{E_3}{E_1} = -\frac{1-nm}{1+nm};$$

well-known optical expressions for the transmitted and reflected amplitudes at perpendicular incidence, except that the possible magnetic property of a transparent medium is usually overlooked.

Treatment of a conducting boundary.—But now, if the medium on the other side of the boundary is a conductor instead of a dielectric, a term in one of the general equations must be modified; and, instead of curl $H = KpE$, we shall have, as the fundamental equation inside the medium,

$$-\frac{dH}{dx} = 4\pi k E;$$

or more generally $(4\pi k + Kp)E$.

So, on the far side of a thin slice of thickness z, the magnetic intensity H_2 is not equal to the intensity $H_1 + H_3$ on the near side, but is less by

$$dH = 4\pi k E dx = 4\pi k E_2 z = 4\pi k \mu v z H_2;$$

and this explains the second of the continuity equations immediately following in the text.

In a quite general case, where all the possibilities of conductivity and capacity &c. are introduced at once, the ratio of E/H is not μv or $(\mu/K)^{\frac{1}{2}}$, but is $(g+\mu p)^{\frac{1}{2}}(4\pi k + Kp)^{-\frac{1}{2}}$ for waves in a general material medium, (g may always be put zero), or $(R+pL)^{\frac{1}{2}}(Q+pS)^{-\frac{1}{2}}$ for waves guided by a resisting wire through a leaky dielectric.

The addition of dielectric capacity to conductivity in a film is therefore simple enough and results in an equation quoted in the text below.

and $v=3\times 10^{10}$ centim. per sec., for v is the velocity in the dielectric not in the conductor; then take a film whose thickness z is one twenty-fifth of a wave-length of the incident light; and the ratio of the transmitted to the incident amplitude comes out

$$1/2\pi\mu k v z=\frac{1000}{\pi v z}=\frac{1}{200}.$$

Some measurements made by W. Wien at Berlin in 1888 (Wied. *Ann.* vol. xxxv.), with a bunsen-burner as source of radiation, give as the actual proportion of the transmitted to the long-wave incident light, for gold whose thickness is

---

Simple treatment of the E.M. theory of light.—It is tempting to show how rapidly the two fundamental electromagnetic equations, in Mr. Heaviside's form, lead to the electromagnetic theory of light, if we attend specially to the direction normal to the plane of the two perpendicular vectors E and H, to the direction along say x, so that $\nabla = i\,d/dx$ and $\nabla^2 = -d^2/dx^2$.

In an insulating medium the equations are

$$\text{curl}\,\mathrm{H}=\mathrm{K}\dot{\mathrm{E}} \qquad \text{and} \qquad -\text{curl}\,\mathrm{E}=\mu\dot{\mathrm{H}};$$

now curl$=\mathrm{V}\nabla=\nabla$, since $\mathrm{S}\nabla=0$ in this case, so

$$\nabla^2\mathrm{H}=\mathrm{K}\nabla\dot{\mathrm{E}}=\mathrm{K}\,\text{curl}\,\dot{\mathrm{E}}=-\mathrm{K}\mu\ddot{\mathrm{H}};$$

or, in ordinary form,

$$\frac{d^2\mathrm{H}}{dx^2}=\mathrm{K}\mu\frac{d^2\mathrm{H}}{dt^2},$$

and there are the waves.

If this is not rigorous, there is no difficulty in finding it done properly in other places. I believe it to be desirable to realize things simply as well.

In a conducting medium the fundamental equations are, one of them,

$$\text{curl}\,\mathrm{H}=\mathrm{K}\dot{\mathrm{E}}+4\pi k\mathrm{E}=(\mathrm{K}p+4\pi k)\mathrm{E},$$

while the other remains unchanged; unless we like to introduce the non-existent auxiliary g, which would make it

$$-\mathrm{V}\nabla\mathrm{E}=(\sigma+\mu p)\mathrm{H},$$

and would cover wires too.

So $$-\nabla^2\mathrm{H}=(4\pi k+\mathrm{K}p)(g+\mu p)\mathrm{H},$$

the general wave equation. In all these equations p stands for d/dt; but, for the special case of simply harmonic disturbance of frequency $p/2\pi$, of course ip can be substituted.

10^{-5} centim., ·0033 or 1/300; while for gold one quarter as thick the proportion was 0·4 (see Appendix II. page 385).

He tried also two intermediate thicknesses, and though approximately the opacity increases with the square of the thickness, it really seems to increase more rapidly: as no doubt it ought, as the boundaries separate. However, for a thickness $\lambda/25$ I suppose we may assume that about 1/3rd of the light would be transmitted, whereas the film-theory gives $(1/200)^2$; so even now a metal calculates out too opaque though it is rather less hopelessly discrepant than it used to be. The result, we see, for the infinitely thin film, is independent of the frequency.

Specific inductive capacity has not been taken into account in the metal, but if it is it does not improve matters. It does not make much difference, unless very large, but what difference it does make is in the direction of increasing opacity. In a letter to me Mr. Heaviside gives for the opacity of a film of highly conducting dielectric

$$\frac{E_1}{E_2} = \left\{ (1+2\pi\mu kvz)^2 + (\tfrac{1}{2}mczp/v)^2 \right\}, \quad . \quad . \quad . \quad (16)$$

where I have replaced his $\frac{1}{2}\mu vz\mathrm{K}p$ last term by an expression with the merely relative numbers K/K_0 and μ/μ_0, called c and m respectively, thus making it easier to realise the magnitude of the term, or to calculate it numerically.

Theory of a Slab.

An ordinary piece of gold-leaf, however, cannot properly be treated as an infinitely thin film; it must be treated as a slab, and reflexions at its boundaries must be attended to. Take a slab between $x=0$ and $x=l$. The equations to be satisfied inside it are the simplified forms of the general fundamental ones

$$-\frac{d\mathrm{E}}{dx} = \mu p\mathrm{H}, \qquad\qquad -\frac{d\mathrm{H}}{dx} = 4\pi k\mathrm{E},$$

k being $1/\sigma$, and K being ignored; while outside, at $x=l$, the condition $\mathrm{E}=\mu v\mathrm{H}$ has to be satisfied, in order that a wave may emerge.

The following solutions do all this if $q^2=4\pi\mu kp$:—

$$E=Ae^{qx}\left(1+\frac{qv+p}{qv-p}e^{2q(l-x)}\right),$$

$$\mu v H=-\frac{Aqv}{p}e^{qx}\left(1-\frac{qv+p}{qv-p}e^{2q(l-x)}\right).$$

Conditions for the continuity of both E and H at $x=0$ suffice to determine A, namely if E_1 H_1 is the incident and E_3 H_3 the reflected wave on the entering side, while E_0 H_0 are the values just inside, obtained by putting $x=0$ in the above,

$$E_1+E_3=E_0,$$
$$E_1-E_3=\mu v(H_1+H_3)=\mu v H_0.$$

Adding, we get a value for A in terms of the incident light E_1,

$$2E_1 p(qv-p)=A\{(qv+p)^2e^{2ql}-(qv-p)^2\}.$$

whence we can write E anywhere in the slab,

$$\frac{E}{E_1}=\frac{2p(qv-p)}{(qv+p)^2e^{2ql}-(qv-p)^2}\left\{e^{qx}+\frac{qv+p}{qv-p}e^{2ql}e^{-qx}\right\}.$$

Put $x=l$, and call the emergent light E_2; then

$$\frac{E_2}{E_1}=\frac{4pqv}{(qv+p)^2e^{ql}-(qv-p)^2e^{-ql}}=\rho\text{, say,} \quad . \quad (17)$$

and this constitutes the measure of the opacity of a slab, ρ^2 being the proportion of incident light transmitted.

It is not a simple expression, because of course p signifies the operator d/dt, and though it becomes simply ip for a simply harmonic disturbance, yet that leaves q complex. However, Mr. Heaviside has worked out a complete expression for ρ^2, which is too long to quote (he will no doubt be publishing the whole thing himself before long), but for slabs of considerable opacity, in which therefore *multiple* reflexions may be neglected, the only important term is

$$\rho=\frac{4\sqrt{2}\,e^{-\alpha l}p/\alpha v}{1+(1+p/\alpha v)^2}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

with

$$\alpha=\sqrt{(2\pi\mu p/\sigma)}=3\times 10^6$$

for light in gold ; and

$$\frac{p}{\alpha v}=\frac{2\pi}{\alpha\lambda}=\frac{2\pi}{30\times 5}=\frac{1}{25}\text{ about.}$$

So the effect of attending to reflexion at the walls of the slab is to still further diminish the amplitude that gets through, below the $e^{-\alpha l}$ appropriate to the unbounded medium, in the ratio of $\frac{2\sqrt{2}}{25}$, or about a ninth.

Effect of each Boundary.

It is interesting to apply Mr. Heaviside's theory to a study of what happens at the first boundary alone, independent of subsequent damping.

Inside the metal, by the two fundamental equations, we have

$$E_0=\left(\frac{\mu p}{4\pi k}\right)^{\frac{1}{2}}H_0,$$

and by continuity across the boundary

$$E_1+E_3=E_0,$$
$$E_1-E_3=\mu v H_0 \quad =\mu v E_0\left(\frac{4\pi k}{\mu p}\right)^{\frac{1}{2}}=\frac{qv}{p}E_0,$$

where still $q^2=4\pi\mu kp$.

Therefore, for the transmitted amplitude

$$\frac{E_0}{E_1}=\frac{2p}{p+qv},$$

and for the reflected

$$\frac{E_3}{E_1}=\frac{p-qv}{p+qv},$$

or rationalising and writing amplitudes only, and understanding by p no longer d/dt in general, but only 2π times the frequency,

$$\frac{E_0}{E_1}=\frac{2p}{\sqrt{((p+\alpha v)^2+\alpha^2 v^2)}}=\rho_1\text{ say.} \quad . \quad (19)$$

Any thickness of metal multiplies this by the factor $e^{-\alpha x}$, and then comes the second boundary, which, according to what has been done above, has a comparatively small but peculiar effect ; for it ought to change the amplitude from ρ_1

into ρ, that is to give an emergent amplitude

$$\frac{4\sqrt{2}\,.\,p/\alpha v}{1+(1+p/\alpha v)^2}\,E_1 e^{-\alpha l},$$

instead of the above incident on the second boundary

$$\frac{2p/\alpha v}{\sqrt{(1+(1+p/\alpha v)^2)}}\,E_1 e^{-\alpha l}\,; \quad . \quad . \quad (20)$$

that is for the case of light in gold, for which $p/\alpha v$ is small, to change $2/\sqrt{2}$ into $2\sqrt{2}$, in other words, to double it.

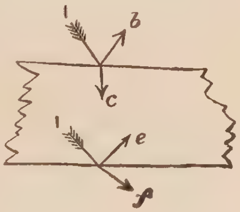
The effect of the first boundary alone, ρ_1, is $\frac{2\sqrt{2\pi}}{\alpha\lambda}$, or say 1/18, and this is a greater reduction effect than that reckoned above for the two boundaries together.

Thus the obstructive effect of the two boundaries together comes out less than that of the first boundary alone—an apparently paradoxical result. About one-eighteenth of the light-amplitude gets through the first boundary, but about one-ninth gets through the whole slab (ignoring the geometrically progressive decrease due to the thickness, that is ignoring $e^{-\alpha l}$, and attending to the effect of the boundaries alone; which, however, cannot physically be done). At first sight this was a preposterous and ludicrous result. The second or outgoing boundary ejects from the medium nearly double the amplitude falling upon it from inside the conductor! But on writing this, in substance, to Mr. Heaviside he sent all the needful answer by next post. "The incident disturbance inside is not the whole disturbance inside."

That explains the whole paradox—there is the reflected beam to be considered too. At the *entering* boundary the incident and reflected amplitudes are in opposite phase, and nearly equal, and their algebraic sum, which is transmitted, is small. At the *emerging* boundary the incident and reflected amplitudes are in the same phase, and nearly equal, and their algebraic sum, which is transmitted, is large—is nearly double either of them. But it is a curious action:—either more light is pushed out from the limiting boundary of a conductor than reaches it inside, or else, I suppose, the velocity of light inside the metal must be greater than it is outside, a result not contradicted by Kundt's refraction experiments, and suggested by most optical

theories. It is worth writing out the slab theory a little more fully, to make sure there is no mistake, though the whole truth of the behaviour of bodies to light can hardly be reached without a comprehensive molecular dispersion theory. I do not think Mr. Heaviside has published his slab theory anywhere yet. A slab theory is worked out by Prof. J. J. Thomson in Proc. Roy. Soc. vol. xlv., but it has partly for its object the discrimination between Maxwell's and other rival theories, so it is not very simple. Lord Kelvin's Baltimore lectures probably contain a treatment of the matter. All that I am doing, or think it necessary to do in an Address, is to put in palatable form matter already to a few leaders likely to be more or less known: in some cases perhaps both known and objected to.

The optical fractions of Sir George Stokes, commonly written $b\ c\ e\ f$, are defined, as everyone knows, as follows. A ray falling upon a denser body with incident amplitude 1 yields a reflected amplitude b and a transmitted c. A ray falling upon the boundary of a rare body with incident amplitude 1 has an internally reflected amplitude e and an emergent f. General principles of reversibility show that $b+e=0$, and that $b^2+cf=1$ in a transparent medium.



Now in our present case we are attending to perpendicular incidence only, and we are treating of a conducting slab; indeed, we propose to consider the obstructive power of the material of the slab so great that we need not suppose that any appreciable fraction of light reflected at the second surface returns to complicate matters at the first surface. This limitation by no means holds in Mr. Heaviside's complete theory, of course, but I am taking a simple case.

The characteristic number which governs the phenomenon is $\frac{p}{av}$ or $\frac{2\pi}{a\lambda}$, a number which for light and gold we reckoned as being about $\frac{1}{27}$, that is decidedly smaller than unity, a being $\sqrt{(2\pi\mu kp)}$ or $\sqrt{\left(\frac{2\pi\mu p}{\sigma}\right)}$. The characteristic number p/av

we will for brevity write as h, and we will express amplitudes for perpendicular incidence only, as follows :—

Incident amplitude 1,

$$\text{externally reflected} \quad b = -\left\{\frac{1+(1-h)^2}{1+(1+h)^2}\right\}^{\frac{1}{2}}$$

$$\text{entering} \quad c = \frac{2h}{\{1+(1+h)^2\}^{\frac{1}{2}}},$$

Incident again 1,

$$\text{internally reflected} \quad e = \left\{\frac{1+(1-h)^2}{1+(1+h)^2}\right\}^{\frac{1}{2}},$$

$$\text{emergent} \quad f = \frac{2\sqrt{2}}{\{1+(1+h)^2\}^{\frac{1}{2}}}.$$

(It must be remembered that e and f refer to the second boundary *alone*, in accordance with the above diagram.)

Thus the amplitude transmitted by the whole slab, or rather by both surfaces together, ignoring the opacity of its material for a moment, is

$$\text{transmitted} \quad cf = \frac{4\sqrt{2}\,.\,h}{1+(1+h)^2}.$$

To replace in this the effect of the opaque material, of thickness l, we have only to multiply by the appropriate exponential damper, so that the amplitude ultimately transmitted by the slab is

$$\frac{4\sqrt{2}\,.\,p/\alpha v}{1+(1+p/\alpha v)^2}e^{-al}$$

times the amplitude originally incident on its front face.

This agrees with the expression (18) specifically obtained above for this case, but, once more I repeat, multiple reflexions have for simplicity been here ignored, and the medium has been taken as highly conducting or very opaque.

But even so the result is interesting, especially the result for f. To emphasize matters, we may take the extreme case when the medium is so opaque that h is nearly zero ; then b is nearly -1, c is nearly 0, being $h\sqrt{2}$, e is the same as b except for sign, and f is nearly 2.

An opaque slab transmits $8h^2e^{-2\alpha l}$ of the incident light energy ; its first boundary transmits only $2h^2$. The second

or emergent boundary doubles the amplitude. Taken in connexion with the facts of selective absorption and the timing of molecules to vibrations of certain frequency, I think that this fact can hardly be without influence on the green transparency of gold-leaf.

APPENDIX I.

MR. HEAVISIDE'S *Note on Electrical Waves in Sea-Water.*

[Contributed to a discussion at the Physical Society in June 1897: see Mr. Whitehead's paper, Phil. Mag. August 1897.]

"To find the attenuation suffered by electrical waves through the conductance of sea-water, the first thing is to ascertain whether, at the frequency proposed, the conductance is paramount, or the permittance, or whether both must be counted.

"It is not necessary to investigate the problem for any particular form of circuit from which the waves proceed. The attenuating factor for plane waves, due to Maxwell, is sufficient. If its validity be questioned for circuits in general, then it is enough to take the case of a simply-periodic point source in a conducting dielectric ('Electrical Papers,' vol. ii. p. 422, § 29). The attenuating constant is the same, viz. (equation (199) *loc cit.*):—

$$n_1=\frac{n}{v}\sqrt{\tfrac{1}{2}}\left[\left\{1+\left(\frac{4\pi k}{cn}\right)^2\right\}^{\frac{1}{2}}-1\right]^{\frac{1}{2}},$$

where $n/2\pi$ is the frequency, k the conductivity, c the permittivity, and $v=(\mu c)^{-\frac{1}{2}}$, μ being the inductivity.

"The attenuator is then $e^{-n_1 r}$ at distance r from the source, as in plane waves, disregarding variations due to natural spreading. It is thus proved for any circuit of moderate size compared with the wave-length, from which simply periodic waves spread.

"The formula must be used in general, with the best values of k and c procurable. But with long waves it is pretty certain that the conductance is sufficient to make $4\pi k/cn$ large. Say with common-salt-solution $k=(30^{11})^{-1}$, then

$$\frac{4\pi k}{cn}=\frac{2k\mu v^2}{f}$$

if f is the frequency. This is large unless f is large, whether we

assume the specific c/c_0 to have the very large value 80 or the smaller value effectively concerned with light waves. We then reduce n_1 to

$$n_1=(2n\mu k\pi)^{\frac{1}{2}}=2\pi(\mu kf)^{\frac{1}{2}},$$

as in a pure conductor.

"This is practically true perhaps even with Hertzian waves, of which the attenuation has been measured in common-salt-solution by P. Zeeman. If then $k^{-1}=30^{11}$ [and if the frequency is 300 per second] we get n_1=about $\frac{1}{5000}$.

"Therefore 50 metres is the distance in which the attenuation due to conductivity is in the ratio 2·718 to 1, and there is no reason why the conductivity of sea-water should interfere, if the value is like that assumed above.

"These formulæ and results were communicated by me to Prof. Ayrton at the beginning of last year, he having enquired regarding the matter, on behalf of Mr. Evershed I believe.

"The doubtful point was the conductivity. I had no data, but took the above k from a paper which had just reached me from Mr. Zeeman. Now Mr. Whitehead uses $k^{-1}=20^{10}$, which is no less than 15 times as great. I presume there is good authority for this datum *. None is given. Using it we obtain $n_1=\frac{1}{1316}$.

"Thus 50 metres is reduced to 13·16 metres. But a considerably greater conductivity is required before it can be accepted that the statements which have appeared in the press, that the failure of the experiments endeavouring to establish telegraphic communication with a light-ship from the sea-bottom was due to the conductance of the sea, are correct. It seems unlikely theoretically, and Mr. Stevenson has contradicted it (in 'Nature') from the practical point of view. So far as I know, no account has been published of these experiments, therefore there is no means of finding the cause of the failure."

* Dr. J. L. Howard has recently set a student to determine the resistivity of the sea-water used by Professor Herdman, density 1·019 gr. per c.c., and he finds it to be 3×10^{10} C.G.S. at 15° C.—O. J. L., March 1899.

Added, May 1899.—An old student of mine, Prof. W. M. Thornton, of the Durham College, Newcastle, tells me that the resistance of sea-water collected at Tynemouth pier at flood-tide is $2{\cdot}39\times10^{10}$ C.G.S. at 15° C.

Appendix II.

Experiments on the Transparency of Metals.

The experiments of W. Wien on the transparency of metals, by means of a bolometer arranged to receive the radiation from a bunsen burner transmitted through different films, resulted in the following numbers for the proportion of radiation transmitted.

Metal.	Thickness in 10^{-7} centim.	Proportion transmitted.		Proportion reflected.
		Bunsen burner luminous.	Bunsen burner non-luminous.	
Platinum	20	·32	·37	·13
Iron & Platinum	40+20	·10	·14	·45
Gold 1	56	·040	·041	·63
Gold 2	100	·0035	·0036	·80
Gold 3	24	·41	·41	·05
Gold 4	35	·20	·20	·19
Silver 1 (blue) ...	36	·058	·046	·78
Silver 2 (grey) ...	39·5	·058	·055	·60
Silver 3 (grey) ...	29	·25	·42	·40
Silver 4 (blue) ...	59·7	·0022	·0019	·95
Silver 5 (grey) ...	27·3	·31	·43	·24

The thickness is in millionths of a millimetre, *i.e.* is in terms of the milli-mikrom * called by microscopists $\mu\mu$.

The films were on glass, and the absorption of the glass was allowed for by control experiments.

* In spelling the word "mikrom" thus, I am following Lord Kelvin, who wishes to introduce two units—one, the ordinary mikrometre (the mikrom) or $\frac{1}{1000}$ of a millimetre, called by microscopists μ; the other, a unit of time to be called a "michron," being the time required for light to traverse the above small distance. This suggestion of Lord Kelvin's commends itself to me; at any rate I take the opportunity of calling the attention of Physicists to it.

It is to be understood that of the whole incident light the proportion reflected is first subtracted, and the residue is then called 1 in order to reckon the fraction transmitted of that which enters the metal, it being understood that the residue which is not transmitted (say ·68 or ·63 in the case of platinum) is absorbed. It may be that more and better work has been done on the opacity of metals than this: at any rate there seems to me room for it. I do not quote these figures with a strong feeling of confidence in their accuracy. They are to be found in Wied. *Ann.* vol. xxxv. p. 57.

Appendix III. (*Added later.*)

The parenthetical statement near the bottom of page 372, concerning the first clear statement of the two circuital laws by Mr. Heaviside, represented subjectively the truth for me and for some others; but it should have been more carefully guarded if it was to be taken as representing objective truth: for I am referred to Maxwell, Philosophical Transactions, 1868 ("Collected Papers," vol. ii. p. 138), where the statement is very clear, and also to Rayleigh, Philosophical Magazine, August 1881, equations 8 and 9. I regret my inadequate acquaintance with scientific history.

XXXVIII. *On certain Diffraction Fringes as applied to Micrometric Observations.* By L. N. G. FILON, *M.A., Demonstrator in Applied Mathematics and Fellow of University College, London**.

1. THE following paper is largely criticism and extension of Mr. A. A. Michelson's memoir "On the Application of Interference Methods to Astronomical Observations," published in the Phil. Mag. vol. xxx. p. 256, March 1891.

Light from a distant source is allowed to pass through two thin parallel slits. The rays are then focussed on a screen (or the retina of the eye) and interference-fringes are seen. If the distant source be really double, or extended, the fringes will disappear for certain values of the distance between the slits. This distance depends on the angle subtended by the two components of the double source or the diameter of the extended source.

Mr. Michelson, however, in obtaining his results treated the breadth of the slits as small compared with the wave-length of light and their length as infinite. This seems unjustifiable *à priori*. The present investigation takes the dimensions of the slits into account.

2. Suppose we have an aperture or diaphragm of any shape in a screen placed just in front of the object-glass of a telescope (fig. 1).

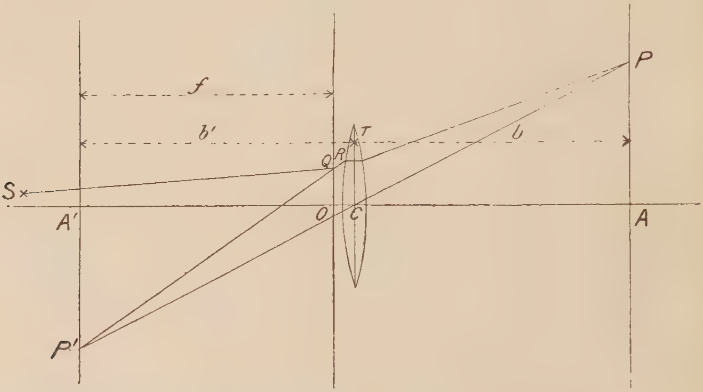
Let the axis of the telescope be the axis of z. Let the axes of x and y be taken in the plane of the diaphragm OQ perpendicular to and in the plane of the paper respectively. Let S be a source of light whose coordinates are U, V, W. Let Q be any point in the diaphragm whose coordinates are (x, y). Let AP be a screen perpendicular to the axis of the telescope, and let (p, q) be the coordinates of any point P on this screen. Let A′P′ be the conjugate image of the screen AP in the object-glass.

Let b = distance of centre C of lens from screen AP.
b' = " " " " image of screen AP.
f = distance of diaphragm OQ from plane A′P′.

* Read November 25, 1898.

Then if, as is usual, we break up a wave of light coming from S at the diaphragm, the secondary wave due to the disturbance at Q would have to travel along a path QRTP in order to reach a point P on the screen, being regularly refracted.

Fig. 1.



But since P′ is the geometrical image of P, all rays which converge to P (*i.e.* pass through P) after refraction, must have passed through P′ before refraction, to the order of our approximation.

Hence the ray through Q which is to reach P must be P′Q.

Moreover, P and P′ being conjugate images the change of phase of a wave travelling from P′ to P is constant to the first approximation and independent of the position of Q.

Now the disturbance at P due to an element $dx\,dy$ of the diaphragm at Q is of the form

$$\frac{A\,dx\,dy}{b\lambda}\sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau}-SQ-QR-\mu\,.\,RT-TP\right),$$

where λ is the wave-length, τ is the period, A is a constant, μ is the index of refraction of the material of the lens, and b is put instead of QP outside the trigonometrical term, because the distance of the lens from the diaphragm and the inclination of the rays are supposed small.

But $\qquad P'Q+QR+\mu RT+TP=\text{constant for P}.$

Therefore the disturbance

$$=\frac{A\,dx\,dy}{b\lambda}\sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau}-SQ+P'Q-\text{const.}\right).$$

But

$$SQ^2=(x-U)^2+(y-V)^2+W^2$$

$$P'Q^2=(x+\frac{b'}{b}p)^2+(y+\frac{b'}{b}q)^2+f^2.$$

Now in practice x, y, p, q are small compared with b', b, f, or W ; U and V are small compared with W. Neglecting terms of order U^3/W^3, xU^2/W^3, &c., we find

$$SQ=\sqrt{U^2+V^2+W^2}-\frac{U}{W}x-\frac{V}{W}y+\frac{1}{2}\frac{x^2+y^2}{W}.$$

In like manner

$$P'Q=b'\sqrt{\frac{f^2}{b'^2}+\frac{p^2+q^2}{b^2}}+\frac{px}{b}+\frac{qy}{b}+\frac{1}{2}\frac{x^2+y^2}{b'},$$

remembering that f is very nearly equal to b' because the diaphragm is very close to the lens.

Hence the difference of retardation measured by length in air

$$P'Q-SQ=\text{const.}+\left(\frac{p}{b}+\frac{U}{W}\right)x+\left(\frac{q}{b}+\frac{V}{W}\right)y$$

$$+\frac{1}{2}(x^2+y^2)\left(\frac{1}{b'}-\frac{1}{W}\right).$$

If now the geometrical image of S lie on the screen AP (*i.e.* if the screen is in correct focus) $b'=W$ and the last term disappears.

If, however, the screen be out of focus $1/b'$ is not equal to $1/W$, and the term in x^2+y^2 may be comparable with the two others, if b' be not very great compared with b. Thus we see that appearances out of focus will introduce expressions of the same kind as those which occur when no lenses are used.

We will, however, only consider the case where the screen is in focus. Let $-u$ and $-v$ be the coordinates of the geometrical image of S ; then

$$u/b=U/W, \qquad v/b=V/W.$$

The difference of retardation measured by length in air is therefore of the form

$$\mathrm{P'Q}-\mathrm{SQ}=\text{const.}+\frac{p+u}{b}x+\frac{q+v}{b}y.$$

Hence the total disturbance at P (integrating over the two slits) is given by the expression

$$\mathrm{D}=\frac{\mathrm{A}}{b\lambda}\int_{a-k}^{a+k}dy\int_{-h}^{h}dx\sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau}+\frac{p+u}{b}x+\frac{q+v}{b}y\right)$$

$$+\frac{\mathrm{A}}{b\lambda}\int_{-a-k}^{-a+k}dy\int_{-h}^{h}dx\sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau}+\frac{p+u}{b}x+\frac{q+v}{b}y\right),$$

where A = a constant,
$2a$ = distance between centres of slits,
$2k$ = breadth of either slit,
$2h$ = length of either slit.

This, being integrated out, gives

$$\mathrm{D}=\frac{2\mathrm{A}b\lambda}{\pi^2(p+u)(q+v)}\sin\frac{2\pi t}{\tau}\cos\frac{2\pi}{\lambda}\frac{q+v}{b}a\sin\frac{2\pi}{\lambda}\frac{q+v}{b}k\sin\frac{2\pi}{\lambda}\frac{p+u}{b}h,$$

whence the intensity of light

$$\iota=\frac{4\mathrm{A}^2b^2\lambda^2}{\pi^4(p+u)^2(q+v)^2}\cos^2\frac{2\pi a}{b\lambda}(q+v)\sin^2\frac{2\pi k}{b\lambda}(q+v)\sin^2\frac{2\pi h}{b\lambda}(p+u)$$

This may be written

$$\frac{64\mathrm{A}^2k^2h^2}{b^2\lambda^2}\cos^2\frac{2\pi a}{b\lambda}(q+v)\frac{\sin^2\frac{2\pi k}{b\lambda}(q+v)\sin^2\frac{2\pi h}{b\lambda}(p+u)}{\left(\frac{2\pi k}{b\lambda}(q+v)\right)^2\left(\frac{2\pi h}{b\lambda}(p+u)\right)^2}.$$

This gives fringes parallel to x and y: k being very small compared with a, the quick variation term in v is

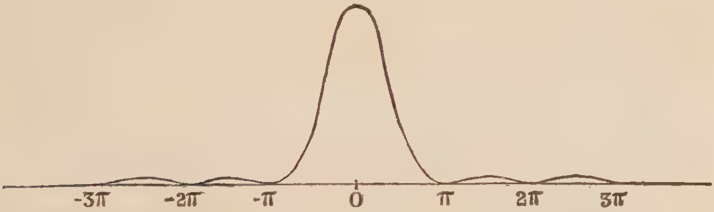
$$\cos^2\frac{2\pi a}{b\lambda}(q+v).$$

Consider the other two factors, namely :

$$\frac{\sin^2 \frac{2\pi k}{b\lambda}(q+v)}{\left\{\frac{2\pi k}{b\lambda}(q+v)\right\}^2} \quad \text{and} \quad \frac{\sin^2 \frac{2\pi h}{b\lambda}(p+u)}{\left\{\frac{2\pi h}{b\lambda}(p+u)\right\}^2}.$$

If we draw the curve $y=\frac{\sin^2 x}{x^2}$ (see fig. 2), we see that these factors are only sensible, and therefore their product is only sensible, for values of $p+u$ and $q+v$ which are numerically less than $b\lambda/2h$ and $b\lambda/2k$ respectively.

Fig. 2.



Hence the intensity becomes very small outside a rectangle whose centre is the geometrical image and whose vertical and horizontal sides are $b\lambda/k$ and $b\lambda/h$ respectively.

This rectangle I shall refer to as the "visible" rectangle of the source.

Inside this rectangle are a number of fringes, the dark lines being given by

$$q+v=\frac{2n+1}{4a}b\lambda$$

and the bright ones by $q+v=nb\lambda/2a$.

The successive maximum and minimum intensities do not vary with a. Hence, what Mr. Michelson calls the measure of visibility of the fringes, namely the quantity

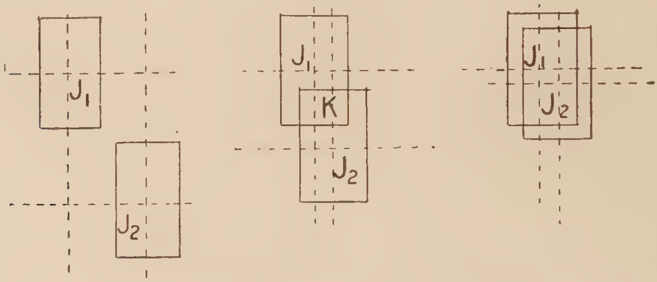
$$\frac{I_1-I_2}{I_1+I_2},$$

where I_1, I_2 are successive maximum and minimum intensities, does not vary with the distance between the slits. The only effect of varying the latter is to make the fringes close up or

open out. Hence for a point-source of light the fringes cannot be made to practically disappear.

3. Consider now two point-sources of light whose geometrical images are J_1, J_2, and draw their visible rectangles (fig. 3).

Fig. 3.



To get the resultant intensity we have to add the intensities at every point due to each source separately.

Then it may be easily seen that the following are the phenomena observed in the three cases shown in fig. 3:—

(1) The two sets of fringes distinct. Consequently no motion of the slits can destroy the fringes. In this case, however, the eye can at once distinguish between the two sources and Michelson's method is unnecessary.

(2) Partial superposition: the greatest effect is round the point K, where the intensities due to the two sources are very nearly equal. If $v'-v$ be the distance between J_1 and J_2 measured perpendicularly to the slits, so that $(v'-v)/b$ is the difference of altitude of the two stars when the slits are horizontal, then over the common area the fringe system is (*a*) intensified if $v'-v$ be an even multiple of $b\lambda/4a$, (*b*) weakened, or even destroyed, if $v'-v$ be an odd multiple of $b\lambda/4a$. For in case (*a*) the maxima of one system are superposed upon the maxima of the other, while in case (*b*) the maxima of the one are superposed upon the minima of the other. This common area, however, will contain only comparatively faint fringes, the more distinct ones round the centres remaining unaffected. We may suppose case (2) to occur whenever the centre of either rectangle lies outside the other, *i. e.* whenever $v'-v > b\lambda/2k$, $u'-u > b\lambda/2h$, $u'-u$ being the horizontal distance between J_1 and J_2.

(3) Almost complete superposition of the visible rectangles. The fringes of high intensity are now affected. These are destroyed or weakened whenever a is an odd multiple of $b\lambda/4(v'-v)$, provided that the intensity of one source be not small compared with that of the other.

Case (3) may be taken to occur when $v'-v<b\lambda/2k$ and $u'-u<b\lambda/2h$.

The smallest value of a for which the fringes disappear is $b\lambda/4(v'-v)$.

If $v'-v$ be very small, this may give a large value of a.

Now a double star ceases to be resolved by a telescope of aperture $2r$ if $(v'-v)/b=\lambda/2r$, and when this relation holds the smallest value of a for which the fringes disappear is not less than $r/2$, which is the greatest separation of the slits which can conveniently be used. Hence the method ceases to be available precisely at the moment when it is most needed.

4. Mr. Michelson, in the paper quoted above, noticed this difficulty, and described an apparatus by means of which the effective aperture of the telescope could be indefinitely increased. He has not shown, however, that the expression for the disturbance remains of the same form, to the order of approximation taken, and he has made no attempt to work out the results when the slit is taken of finite width, as it should be.

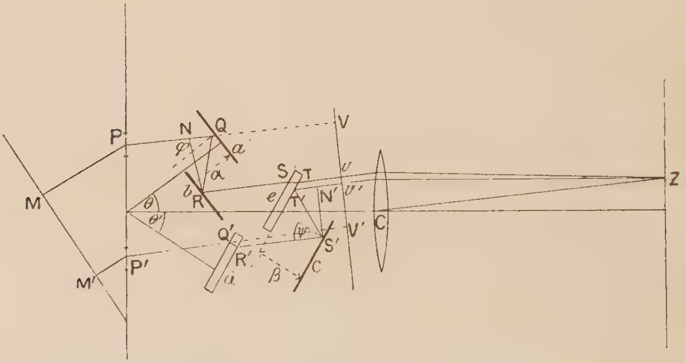
In his paper Mr. Michelson describes two kinds of apparatus. I shall confine my attention to the second one, as being somewhat more symmetrical.

So far as I can gather from Mr. Michelson's description, the instrument consists primarily of a system of three mirrors a, b, c and two strips of glass e, d (fig. 4). The mirrors a and b are parallel, and c, d, e are parallel. Light from a point P in one slit is reflected at Q and R by the mirrors a and b, is refracted through the strip e, and finally emerges parallel to its original direction as T U. Light from a point P′ in the other slit is refracted through the strip d, and reflected at S′ and T′ by the strips e and c.

I may notice in passing that the strip e should be half silvered, but *not at the back*, for if the ray S′ T′ is allowed to penetrate inside the strip and emerge after two refractions

and one reflexion, not only is a change of phase introduced, owing to the path in the glass, which complicates the analysis, but the conditions of reflexion, which should be the same for all four mirrors, are altered, and this changes the intensities of the two streams. We shall see afterwards that this silvering can be done without impeding the passage of the transmitted stream, as it will turn out that the two streams must be kept separate.

Fig. 4.



Suppose then that a plane wave of light whose front is M M′ is incident upon the diaphragm. Let us break the wave up, as is usual, in the plane of the diaphragm. Let Z be a point on the screen whose cordinates are (p, q) at which the intensity of light is required.

Then if C be the centre of the object-glass, the direction in which rays T U, T′ U′ must proceed in order to converge to Z after refraction is parallel to C Z.

Hence PQ, RS, TU, P′Q′, R′S′, T′U′ are all parallel to CZ, and the direction-cosines of CZ are

$$\frac{p}{\sqrt{p^2+q^2+b^2}}, \quad \frac{q}{\sqrt{p^2+q^2+b}}, \quad \frac{b}{\sqrt{p^2+q^2+b^2}}.$$

I shall assume that the strips e and d are cut from the same plate and are of equal thickness. This will sensibly simplify the analysis, though, as I think, it would not materially influence the appearances if the strips were unequal.

If, however, we suppose them equal, we may neglect the

presence of strips, as far as refraction is concerned, since clearly the retardation introduced is the same for all parallel rays.

If now UU′ be a plane perpendicular to CZ, then, since we know that rays parallel to TU, T′U′ converge to a focus at Z, the only parts of the paths of the rays which can introduce a difference of phase are

$$\text{MP}+\text{PQ}+\text{QR}+\text{RS}+\text{TU for one stream,}$$
$$\text{M}'\text{P}'+\text{P}'\text{Q}'+\text{R}'\text{S}'+\text{S}'\text{T}'+\text{T}'\text{U}' \text{ for the other stream.}$$

Produce PQ, P′Q′ to meet it in V and V′ and let N, N′ be the feet of the perpendiculars from R and S′ on PQ, T′U′ respectively.

Thus we may take the change of phase as due to the retardation

$$(\text{MP}+\text{PV})+(\text{NQ}+\text{QR})$$

for diffraction at one slit, and to the retardation

$$(\text{M}'\text{P}'+\text{P}'\text{V}')+(\text{S}'\text{T}'+\text{T}'\text{N}')$$

for rays proceeding from the other slit.

The terms in the first brackets give us the expression which we had before, viz. :—

$$-\left(\frac{p+u}{b}\right)x-\left(\frac{q+v}{b}\right)y+\text{const.}$$

As to the other terms

$$\text{NQ}+\text{QR}=\text{QR}\ (1+\cos 2\phi)=\alpha\frac{1+\cos 2\phi}{\cos\phi}=2\alpha\cos\phi,$$

where α is the distance between the mirrors a, b and ϕ is the angle of incidence of any ray on these mirrors.

Similarly $\text{S}'\text{T}'+\text{T}'\text{N}'=2\beta\cos\psi$, where β is the distance between e and c and ψ the angle of incidence of any ray upon e and c.

Now if the mirrors a and b are inclined to the plane of the diaphragm at an angle θ, c, d, e at an angle $(-\theta')$, then

$$\cos\phi=\frac{q\sin\theta+b\cos\theta}{\sqrt{p^2+q^2+b^2}},$$

$$\cos\psi=\frac{-q\sin\theta'+b\cos\theta'}{\sqrt{p^2+q^2+b^2}}.$$

To find the disturbance at Z we have

$$\int_{a-k}^{a+k} dy \int_{-h}^{+h} dx \frac{A}{b\lambda} \sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau} + \frac{p+u}{b}x + \frac{q+v}{b}y - 2\alpha\cos\phi\right)$$

$$+ \int_{-a-k}^{-a+k} dy \int_{-h}^{+h} dx \frac{A}{b\lambda} \sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau} + \frac{p+u}{b}x + \frac{q+v}{b}y - 2\beta\cos\psi\right),$$

which, on being integrated, gives

$$\frac{2Ab\lambda}{\pi^2(p+u)(q+v)} \sin\frac{2\pi}{\lambda}\frac{q+v}{b}k \sin\frac{2\pi}{\lambda}\frac{p+u}{b}h \sin\frac{2\pi}{\lambda}\left(\frac{\lambda t}{\tau} - \epsilon\right)\cos\frac{2\pi\gamma}{\lambda},$$

where $-\epsilon-\gamma=-\frac{v+q}{b}a-2\beta\cos\psi$, $\quad -\epsilon+\gamma=\frac{v+q}{b}a-2\alpha\cos\phi$,

whence $\quad \epsilon=\alpha\cos\phi+\beta\cos\psi,$

$$\gamma=\frac{v+q}{b}a+\beta\cos\psi-\alpha\cos\phi.$$

Hence the intensity I of light at (p, q) is

$$\frac{4A^2b^2\lambda^2}{\pi^4(p+u)^2(q+v)^2} \sin^2\frac{2\pi}{\lambda}\frac{q+v}{b}v \sin^2\frac{2\pi}{\lambda}\frac{p+u}{b}h \cos^2\frac{2\pi\gamma}{\lambda}$$

where $\gamma=\frac{v+q}{b}a+\frac{\beta\cos\theta'-\alpha\cos\theta}{\sqrt{p^2+q^2+b^2}}b-\frac{(\beta\sin\theta'+\alpha\sin\theta)}{\sqrt{p^2+q^2+b^2}}q.$

In the last term we may put $\frac{q}{\sqrt{p^2+q^2+b^2}}=\frac{q}{b}$, for if we went to a higher approximation, we should introduce *cubes* of p/b, q/b which we have hitherto neglected.

If, further, we make $\beta\cos\theta'=\alpha\cos\theta$, which can always be managed without difficulty, the second term, which would contain squares on expansion, disappears and we have

$$\gamma=\frac{(v+q)a-(\beta\sin\theta'+\alpha\sin\theta)q}{b}$$

$$=\{va+q(a-(\beta\sin\theta'+\alpha\sin\theta))\}/b.$$

This gives fringes of breadth $b\lambda/2(a-(\beta\sin\theta'+a\sin\theta))$. These may be reckoned from the bright fringe $\gamma=0$; *i. e.*

$$q_0=\frac{-va}{a-(\beta\sin\theta'+\alpha\sin\theta)}.$$

The visibility of the fringes for a single source will, as before, not be affected by changing a: for a second source the origin of the fringes is given by

$$q_0' = -v'a/(a - \{\beta \sin \theta' + \alpha \sin \theta\}).$$

and if the visible rectangles overlap, there will be a sensible diminution of the fringe appearance whenever

$$q_0 - q_0' = (n + \tfrac{1}{2}) b\lambda / 2\{a - (\beta \sin \theta' + \alpha \sin \theta)\}$$

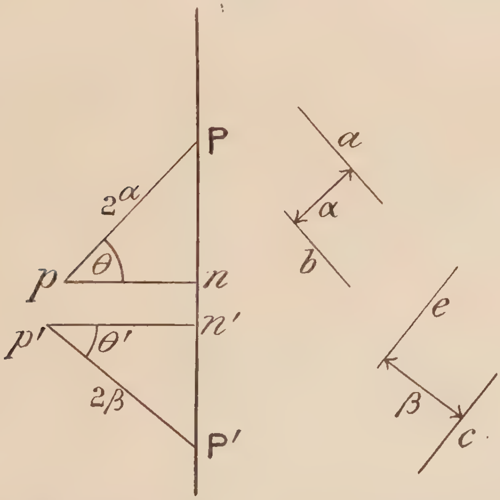
where n is an integer;

i.e. $v' - v =$ an odd multiple of $b\lambda/4a$,

the condition previously found.

One further point should be noticed: if a be very nearly equal to $\beta \sin \theta' + \alpha \sin \theta$ the fringes become too broad to be observed, whatever the source may be.

Fig. 5.



To see the physical meaning of this condition, and also of the condition $\beta \cos \theta' = \alpha \cos \theta$, we notice that a point source of light P at the centre of one of the slits appears after reflexion at the two mirrors a, b, to be at p, where Pp is equal to twice the distance between the mirrors and is perpendicular to their plane (fig. 5). Hence the double reflexion removes the image of the slit a distance $2\alpha \cos \theta$ behind the diaphragm and $2\alpha \sin \theta$ closer to the centre. In the same way the image of

the other slit is brought $2\beta\cos\theta'$ behind the diaphragm and $2\beta\sin\theta'$ nearer the centre.

Our condition $\beta\cos\theta'=a\cos\theta$ therefore means that the images of the two slits must be in the same plane parallel to the plane of the diaphragm itself, and our second condition shows that they must be some distance apart.

To find the minimum of this distance, remember that the fringes will be invisible if the distance between successive maxima exceeds the vertical dimension of the visible rectangle: in other words, if

$$b\lambda/2(a-(\beta\sin\theta'+\alpha\sin\theta))>b\lambda/k,$$

or distance in question $<k$,

which means that the centre of the image of either slit must be outside the other. These two points must be carefully borne in mind in adjusting the instruments.

When this, however, is done, we see that Michelson's assertions are confirmed, and that when we increase the aperture of the telescope in this way, the results obtained are of the same character as when the slits are placed directly in front of the object-glass.

5. Let us now proceed to consider an extended source, which we shall suppose for simplicity to be of uniform intensity.

The intensity at a point (p, q) on the screen will be of the form

$$I=A^2\frac{64h^2k^2}{b^2\lambda^2}\iint\left\{\frac{\sin\dfrac{2\pi k(q+v)}{b\lambda}}{\left(\dfrac{2\pi k(q+v)}{b\lambda}\right)}\cdot\frac{\sin\dfrac{2\pi h(p+u)}{b\lambda}}{\left(\dfrac{2\pi h(p+u)}{b\lambda}\right)}\right\}^2\cos^2\frac{2\pi a(q+v)}{v\lambda}\,du\,dv$$

the integral being taken all over the geometrical image of the extended source.

We have now three cases to consider.

(a) When the angular dimensions of the source are large compared with λ/h.

(b) When the angular dimensions of the source are small compared with λ/h.

(c) When the angular dimensions of the source are neither large nor small compared with λ/h.

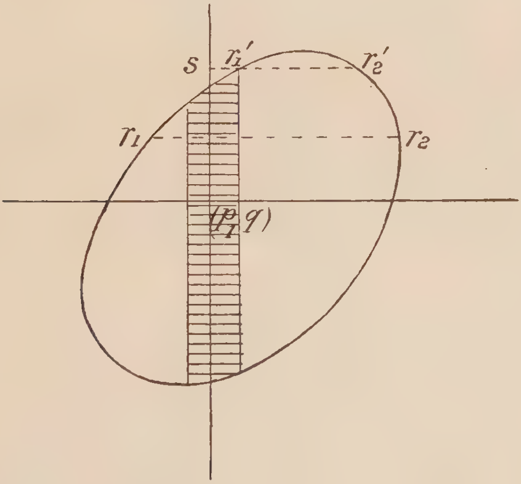
Let us begin with case (a). Then, if we consider a point inside the geometrical image, the two limits for u will be very large, except where the vertical through the point cuts the image ; the quantity

$$\left\{\frac{\sin\frac{2\pi h(p+u)}{b\lambda}}{\frac{2\pi h(p+u)}{b\lambda}}\right\}^2$$

being insensible for all points outside a thin strip (shaded in the figure) having for its central line the line through p, q perpendicular to the slits.

We may therefore, in integrating with regard to u, replace the limits by $\pm\infty$, and then integrate with regard to v along the chord of the image perpendicular to the slits.

Fig. 6.



Hence, remembering that

$$\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi,$$

it follows that

$$\mathrm{I}=\frac{32\,A^2hk^2}{b\lambda}\int\left\{\frac{\sin\frac{2\pi k}{b\lambda}(q+v)}{\frac{2\pi k}{b\lambda}(q+v)}\right\}^2\cos^2\frac{2\pi a}{b\lambda}(q+v)\,dv.$$

Now if the angular dimensions of the source of light be large compared with λ/k, the limits of integration with regard to v may be made infinite. In this case the intensity I

$$= \frac{8A^2hk^2}{b\lambda}\int_{-\infty}^{\infty} dv \left\{\sin^2\frac{2\pi}{b\lambda}(a+k)(q+v)+\sin^2\frac{2\pi}{b\lambda}(a-k)(q+v)\right.$$

$$\left.-2\sin^2\frac{2\pi a}{b\lambda}(q+v)+2\sin^2\frac{2\pi k}{b\lambda}(q+v)\right\} \div \left(\frac{2\pi k(q+v)}{b\lambda}\right)^2$$

$$= \frac{4A^2h}{\pi}\left[\{(a+k)+(a-k)-2a+2k\}\int_{-\infty}^{+\infty}\frac{\sin^2 x}{x^2}dx\right]$$

$$= 8A^2hk = \text{constant}.$$

This result shows us that if the dimensions of the source exceed a certain limit, no diffraction-fringes exist at all, at least near the centre of the image. Next let the angular dimensions of the source be less than $\frac{\lambda}{12k}$, then throughout the integration $\frac{2\pi k}{b\lambda}(q+v)$ is less than $\pi/6$ numerically.

But

$$\frac{\sin^2\frac{\pi}{6}}{\pi^2/36} = \frac{9}{\pi^2},$$

and differs but little from unity.

We may therefore in this case write

$$\frac{\sin^2\frac{2\pi k}{b\lambda}(q+v)}{\left(\frac{2\pi k}{b\lambda}(q+v)\right)^2} = 1$$

throughout the range of integration.

If now the limits be v_1 and v_2 we have

$$I = \frac{32A^2hk^2}{b\lambda}\int_{v_2}^{v_1}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos\frac{4\pi a(q+v)}{b\lambda}\right)dv$$

$$= \frac{4A^2hk^2}{\pi a}\int_{v_2}^{v_1}\left(1+\cos\frac{4\pi a(q+v)}{b\lambda}\right)d\left(\frac{4\pi a(q+v)}{b\lambda}\right)$$

$$= \frac{4A^2hk^2}{\pi a}\left\{\frac{4\pi a}{b\lambda}(v_1-v_2)+\sin\frac{4\pi a(q+v_1)}{b\lambda}-\sin\frac{4\pi a(q+v_2)}{b\lambda}\right\}$$

$$= \frac{4A^2hk^2}{\pi a}\left\{\frac{4\pi a(v_1-v_2)}{b\lambda}+2\sin\frac{2\pi a(v_1-v_2)}{b\lambda}\cos\frac{4\pi a(q+\frac{1}{2}(v_1+v_2)}{b\lambda}\right.$$

Let $2c$=length of chord through the point perpendicular to the direction of the slits, then

$$2c=v_1-v_2,$$

and let v_0=coordinate of the mid-point of this chord. Then

$$\mathrm{I}=\frac{8\mathrm{A}^2hk^2}{\pi a}\left\{\frac{4\pi ac}{b\lambda}+\sin\frac{4\pi ac}{b\lambda}\cos\frac{4\pi a(q+v_0)}{b\lambda}\right\}.$$

The fringes therefore disappear when

$$\frac{4\pi ac}{b\lambda}=s\pi.$$

Their visibility is

$$\sin\frac{4\pi ac}{b\lambda}\Big/\frac{4\pi ac}{b\lambda},$$

and is a maximum when

$$\frac{4\pi ac}{b\lambda}=\tan\frac{4\pi ac}{b\lambda};$$

but the most visible fringes correspond to the early maxima.

This form agrees exactly with the formula given by Mr. Michelson for a uniformly illuminated segment of a straight line perpendicular to the slits. We see, however, that, provided the conditions stated be fulfilled, it is applicable to a source of any shape.

The most general form of the fringes is given by

$$q+\tfrac{1}{2}(v_1+v_2)=\text{const.},$$

and therefore consists of lines parallel to the locus of middle points of chords at right angles to the slits. These will be straight lines in the case of a rectangular, circular, or elliptic source. Here, however, a new difficulty presents itself. For the rectangular source v_1-v_2 will be constant, whatever chord perpendicular to the slits we may select. Fringes will therefore appear and disappear as a whole.

But for a circular or elliptic source, v_1-v_2 *varies* as we pass from chord to chord. Thus the maxima will be invisible for some chords when they are most visible for others and conversely. Hence, whatever be the distance between the slits, it appears at first as if we might always expect a mottled appearance.

But in the case of a circular or elliptic source the length of

the chord varies extremely slowly *near the centre* and *there* fringes will be visible, the length of the chord being practically constant. The mottled appearance, on the other hand, will predominate as we approach the sides.

6. Consider now case (b) and let the dimensions of the source be so small that, for any point sufficiently close to the centre of the image $\dfrac{2\pi h(p+u)}{b\lambda}$ is a small angle throughout the range of integration.

[For points not near the centre of the image the illumination will be very small and the appearances are comparatively unimportant.]

For a point distant $<\dfrac{\lambda b}{24h}$ from the centre of the image, we may put, as in previous reasoning,

$$\left\{\frac{\sin\dfrac{2\pi k}{b\lambda}(q+v)}{\dfrac{2\pi k}{b\lambda}(q+v)}\;\frac{\sin\dfrac{2\pi h}{b\lambda}(p+u)}{\dfrac{2\pi h}{b\lambda}(p+u)}\right\}^2=1$$

all over the range of integration, whence

$$\mathrm{I}=\frac{32\mathrm{A}^2k^2h^2}{b^2\lambda^2}\iint\left(1+\cos\frac{4\pi a(q+v)}{b\lambda}\right)du\,dv$$

$$=\frac{32\mathrm{A}^2k^2h^2}{b^2\lambda^2}\left(\Omega+\cos\frac{4\pi aq}{b\lambda}\int u\cos\frac{4\pi av}{b\lambda}\,dv-\sin\frac{4\pi aq}{b\lambda}\int u\sin\frac{4\pi av}{b\lambda}\,d\right.$$

$$=\frac{32\mathrm{A}^2k^2h}{b^2\lambda^2}\left(\Omega+\mathrm{R}\cos\left(\frac{4\pi aq}{b\lambda}+\phi\right)\right),$$

where $\quad\Omega=$ total area of the image,

$$\mathrm{R}\cos\phi=\int u\cos\frac{4\pi av}{b\lambda}\,dv,\quad \mathrm{R}\sin\phi=\int u\sin\frac{4\pi av}{b\lambda}\,dv,$$

the integrals being taken all over the image. The visibility $=\mathrm{R}/\Omega$ and therefore vanishes when R vanishes.

In the case of a circular source we find

$$\phi=0,\qquad \mathrm{R}=(\text{some non-vanishing factor}),\ \mathrm{J}_1\left(\frac{4\pi ar}{b\lambda}\right),$$

where J_1 is the Bessel's function of order unity and r is the radius of the image, so that r/b is the angular radius of the source. The dark fringes are given by

$$\frac{4\pi aq}{b\lambda}=(2s+1)\pi,$$

q being measured from the centre of the source. The fringes are parallel to the slits and disappear whenever

$$J_1\left(\frac{4\pi ar}{b\lambda}\right)=0.$$

This result agrees with the one given by Michelson for any circular source. We see that it only holds provided the dimensions of the source do not exceed a certain limit.

In the case of an elliptic source $\phi=0$ also, and R is not altered by any sliding of the image parallel to the direction of the slits. Hence we may replace the oblique ellipse by one having its principal axis parallel and perpendicular to the slits, the values of the semi-axes being d and ϖ, where $d=$ length of semi-diameter of original image parallel to the slits, $\varpi=$ length of perpendicular from the centre upon the parallel tangent. We find without difficulty:

$$\mathrm{R}=\int\cos\frac{4\pi av}{b\lambda}udv \text{ over the image}$$

$$=4\int_0^{\varpi}\frac{d}{\varpi}\sqrt{\varpi^2-v^2}\cos\frac{4\pi av}{b\lambda}dv=\frac{d\varpi}{2}\cdot\frac{b\lambda}{a\varpi}J_1\left(\frac{4\pi a\varpi}{b\lambda}\right)$$

$$=\Omega\left(\frac{b\lambda}{2\pi a\varpi}\right)J_1\left(\frac{4\pi a\varpi}{b\lambda}\right).$$

The visibility is therefore

$$2\,J_1\left(\frac{4\pi a\varpi}{b\lambda}\right)\Big/\left(\frac{4\pi a\varpi}{b\lambda}\right),$$

and vanishes whenever

$$J_1\left(\frac{4\pi a\varpi}{b\lambda}\right)=0.$$

Hence we see that, for an elliptic source, if $\rho=$length of semi-diameter perpendicular to the slits, $\varpi=$length of perpendicular on the tangent parallel to the slits, then the fringes disappear

when $\sin\frac{4\pi a\rho}{b\lambda}=0$, if the angular dimensions are of order $\frac{1}{12}\frac{\lambda}{k}$ as indicated above, and when $J_1\left(\frac{4\pi a\varpi}{b\lambda}\right)=0$, when the angular dimensions are less than $\frac{1}{12}\frac{\lambda}{h}$.

In the first case ρ is the quantity which determines the disappearance of the fringes, in the second case ϖ: and further, we see that the validity of the formulæ is entirely dependent on the *length* and *breadth* of the slit, neither of which is considered by Mr. Michelson.

We may notice that the best results are obtained, in the first case when h is large, in the second case when h is small.

7. It remains to consider the intermediate case (c). This does not perhaps present so much interest as the other two; the first will generally correspond to the case of a planet, the second to that of a star, in astronomical observations.

In dealing with case (c) we shall suppose the angular dimensions to be small, with regard to λ/k, but not with regard to λ/h. We may then write

$$\mathrm{I}=\frac{64\mathrm{A}^2h^2k^2}{b^2\lambda^2}\iint\left\{\frac{\sin\frac{2\pi k(q+v)}{b\lambda}}{\frac{2\pi k(q+v)}{b\lambda}}\cdot\frac{\sin\frac{2\pi h(p+u)}{b\lambda}}{\frac{2\pi h(p+u)}{b\lambda}}\right\}^2\cos^2\frac{2\pi a(q+v)}{b\lambda}\,d\ldots$$

$$=\frac{32\mathrm{A}^2h^2k^2}{b^2\lambda^2}\iint\frac{\sin^2\frac{2\pi hu}{b\lambda}}{\left(\frac{2\pi hu}{b\lambda}\right)^2}\left(1+\cos\frac{4\pi a(q+v)}{b\lambda}\right)du\,dv,$$

if we only consider the appearances along the line $p=0$, taken to pass through the centre of the image, which is assumed circular or elliptic.*

Denoting $\frac{2\pi h}{b\lambda}$ by μ and expanding, we get

$$\frac{\sin^2\frac{2\pi hu}{b\lambda}}{\left(\frac{2\pi hu}{b\lambda}\right)^2}=1-\tfrac{1}{2}\cdot\frac{2^2\mu^2u^2}{3!}+\ldots+\frac{(-1)^{r-1}(2\mu u)^{2(r-1)}}{r\,.\,(2r-1)!}+\ldots,$$

* In the case of the elliptic source, the line is not, strictly speaking, $p=0$, but is the diameter conjugate to the direction of the slits, and the same formula holds.

whence

$$=\frac{32A^2h^2k^2}{b^2\lambda^2}\Big[K+\int\Big(u-\frac{1}{2.3}\frac{(2\mu)^2u^3}{3!}+\ldots+\frac{(-1)^{r-1}(2\mu)^{2(r-1)}}{r(2r-1)}\frac{u^{2r-1}}{(2r-1)!}+\ldots\Big)\times\cos\frac{4\pi a(q+v)}{b\lambda}dv\Big]$$

where

$$K=\iint\frac{\sin^2\mu u}{\mu^2u^2}dudv$$

taken all over the source, and is essentially positive and independent of a and q.

Now

$$\int u^{2r-1}\sin\frac{4\pi a(v)}{b\lambda}dv=0$$

for a circle or ellipse.

To find the cosine integral, C, we have, d and ϖ having the same meaning as before,

$$C=4\int_0^{\varpi}\frac{d^{2r-1}}{\varpi^{2r-1}}(\varpi^2-v^2)^{r-\frac{1}{2}}\cos\frac{4\pi av}{b\lambda}dv.$$

Put $v=\varpi\cos\theta$.

$$C=4d^{2r-1}\varpi\int_0^{\frac{\pi}{2}}\sin^{2r}\theta\cos\left(\frac{4\pi a\varpi}{b\lambda}\cos\theta\right)d\theta$$

$$=\frac{2d^{2r-1}\varpi 2^r\sqrt{\pi}\Gamma(r+\frac{1}{2})}{\left(\frac{4\pi a\varpi}{b\lambda}\right)^r}J_r\left(\frac{4\pi a\varpi}{b\pi}\right)$$

$$=\frac{2d^{2r-1}\varpi\pi(2r-1)!}{2^{r-1}(r-1)!}\frac{J_r\left(\frac{4\pi a\varpi}{b\lambda}\right)}{\left(\frac{4\pi a\varpi}{b\lambda}\right)^r}.$$

Hence:

$$=\frac{32A^2h^2k^2}{b^2\lambda^2}\left\{K+\cos\frac{4\pi aq}{b\lambda}\left(\sum_{r=1}^{r=\infty}\frac{2\pi d^{2r-1}\varpi(-1)^{r-1}}{\left(\frac{4\pi a\varpi}{b\lambda}\right)^r}\frac{\left(\frac{4\pi h}{b\lambda}\right)^{2r-2}}{r(2r-1)}\frac{J_r\left(\frac{4\pi a\varpi}{b\lambda}\right)}{2^{r-1}(r-1)!}\right)\right\}$$

$$=\frac{32A^2h^2k^2}{b^2\lambda^2}\left[K+2\cos\frac{4\pi aq}{b\lambda}\frac{\pi\varpi d}{\left(\frac{4\pi a\varpi}{b\lambda}\right)}\sum_{r=1}^{r=\infty}(-1)^{r-1}\left\{\frac{\left(\frac{4\pi hd}{b\lambda}\right)^2}{\frac{8\pi a\varpi}{b\lambda}}\right\}^{r-1}\frac{J_r\left(\frac{4\pi a\varpi}{b\lambda}\right)}{r!(2r-1)}\right].$$

Denote by Ω the total area of the image and by e the ratio

$$\left(\frac{4\pi hd}{b\lambda}\right)^2 \Big/ \left(\frac{8\pi a\varpi}{b\lambda}\right).$$

Then the visibility

$$= \frac{2\Omega}{\mathrm{K}} \frac{1}{\frac{4\pi a\varpi}{b\lambda}} \left\{ \mathrm{J}_1\left(\frac{4\pi a\varpi}{b\lambda}\right) - \frac{e}{6}\mathrm{J}_2\left(\frac{4\pi a\varpi}{b\lambda}\right) + \ldots \right.$$
$$\left. + (-1)^{r-1}\frac{e^{r-1}}{r\,!\,(2r-1)}\mathrm{J}_r\left(\frac{4\pi a\varpi}{b\lambda}\right) + \ldots \right\}$$

The series for the visibility is absolutely convergent, because $\mathrm{J}_{n+1}(x)/\mathrm{J}_n(x)$ decreases numerically without limit as n increases without limit.

The roots of the equation

$$\mathrm{J}_1\left(\frac{4\pi a\varpi}{b\lambda}\right) - \frac{e}{6}\mathrm{J}_2\left(\frac{4\pi a\varpi}{b\lambda}\right) + \ldots + (-1)^{r-1}\frac{e^{r-1}}{r\,!\,(2r-1)}\mathrm{J}_r\left(\frac{4\pi a\varpi}{b\lambda}\right) + \ldots = 0$$

give the values of a for which the visibility vanishes.

Notice, however, that e contains d and the length of the slit, so that the values obtained for a will be functions of the horizontal diameter and of the length of the slit.

8. Besides enabling us to determine the angular distance of two point-sources and the radius of an extended source, Mr. Michelson's method allows us to detect and measure the ellipticity of a luminous disk.

Referring to the formulæ for cases (a) and (b), the visibility vanishes when

$$\sin\frac{4\pi a\rho}{b\lambda} = 0 \text{ in case } (a),$$

and when

$$\mathrm{J}_1\left(\frac{4\pi a\varpi}{b\lambda}\right) = 0 \text{ in case } (b).$$

In either of these cases, if we rotate the slits about the axis of the telescope, without altering a, then if the source is elliptic, ρ and ϖ will vary, and the visibility of the fringes will vary.

Now suppose for a given position of the slits we vary a until the visibility$=0$ for that position, and then rotate the slits and note the different inclinations for which it vanishes.

It will certainly vanish once again before a complete half-

turn has been made, namely, when the slits make an angle with the direction of either axis of the ellipse equal to that which they made at first, but on the other side of the axis.

It may vanish more than once, but since the inclinations for which it vanishes are symmetrical with regard to the axes of the ellipse, there will usually be no difficulty in determining the *directions* of the axes.

Their *lengths* can then be determined by two observations of the disappearance of the fringes, one for each of the two positions of the slits which are perpendicular to an axis.

It must, however, be noticed that the accuracy of this method for measuring ellipticity decreases with the size of the source, inasmuch as the quantity which causes the alteration in the fringes is the *difference*, not the *ratio* of the semi-axes.

To get some idea of the sensitiveness of the method, let us estimate roughly the amount of ellipticity which could be detected in a disk of angular semi-diameter $10''$, taking the mean wave-length of light $\cdot 5 \times 10^{-4}$ cms.

The visibility vanishes when $\sin\frac{4\pi a\rho}{b\lambda} = 0$, and will be quite sensible when $\sin\frac{4\pi a\rho}{b\lambda} = \frac{1}{2}$, say. Hence in order that we may be able to note a sensible difference of visibility in the fringes, we must have

$$\frac{4\pi a}{\lambda}\left(\frac{\rho_1 - \rho_2}{b}\right) = \frac{\pi}{6} \text{ at least;}$$

or

$$\frac{\rho_1 - \rho_2}{b} = \frac{1}{2}\,\frac{1}{10^6}$$

if a be a little above 4 cms.

$\therefore$ difference of angular semi-axes $= \cdot 01$ (semi-diameter) *q.p.*, or the amount of ellipticity which can be detected $= \cdot 01$.

I have taken $\sin\frac{4\pi a\rho}{b\lambda} = 0$ as giving zero visibility, because this example will clearly fall under case (a).

9. Summing up the results obtained we see that:—

(1) It is possible by the observation of Michelson's interference-fringes to separate a double point-source, or detect breadth and ellipticity in a slightly extended source.

(2) But the distance between the two points, or the dimensions of the extended source, must lie within certain limits depending on the length and breadth of the slits *.

(3) The dimensions of the slits also considerably affect the general theory, the formulæ obtained not being identical with Michelson's. The law of appearance and disappearance of the fringes depends very largely on the distance between the points or the dimensions of the extended source.

Discussion.

Prof. S. P. Thompson asked what was the minimum width of slit with which the author had found it practicable to work.

In reply, the Author said that with monochromatic light the minimum width he was able to use with his telescope was about half a millimetre.

* Since the above was read, a paper has appeared in the *Comptes Rendus de l'Académie des Sciences* for Nov. 28, 1898, dealing with the modifications in Michelson's formulæ when we take into account the breadth of the slits. The author, M. Hamy, follows Michelson in not considering variations of intensity parallel to the slits. This, I think, accounts for his results not quite agreeing with mine.

XXXIX. *The Equivalent Resistance and Inductance of a Wire to an Oscillatory Discharge.* By EDWIN H. BARTON, *D.Sc., F.R.S.E., Senior Lecturer in Physics, University College, Nottingham* *.

IN an article in the Philosophical Magazine for May 1886 †, Lord Rayleigh, whilst greatly extending Maxwell's treatment of the self-induction of cylindrical conductors, confined the discussion of alternating currents to those which followed the harmonic law with constant amplitude. The object of the present note is to slightly modify the analysis so as to include also the decaying periodic currents obtained in discharging a condenser and the case of the damped trains of high-frequency waves generated by a Hertzian oscillator and now so often dealt with experimentally. In fact, it was while recently working with the latter that the necessity of attacking this problem occurred to me.

Résumé of previous Theories.—To make this paper intelligible without repeated references to both Maxwell and Rayleigh, it may be well to explain again the notation used and sketch the line of argument followed.

The conducting wire is supposed to be a straight cylinder of radius a, the return wire being at a considerable distance. The vector potential, H, the density of the current, w, and the "electromotive force at any point" may thus be considered as functions of two variables only, viz., the time, t, and the distance, r, from the axis of the wire. The total current, C, through the section of the wire, and the total electromotive force, E, acting round the circuit, are the variables whose relation is to be found. It is assumed that

$$H = S + T_0 + T_1 r^2 + \dots + T_n r^{2n}, \quad . \quad . \quad . \quad (1)$$

where S, T_0, T_1, &c., are functions of the time. A relation between the T's is next established so that the subscripts are replaced by coefficients. The value of H at the surface of the

* Read January 27, 1899.

† "On the Self-Induction and Resistance of Straight Conductors."

wire is equated to AC, where A is a constant. This leads to Maxwell's equation (13) of art. 690. The magnetic permeability, μ, of the wire, which Maxwell had treated as unity, is now introduced by Lord Rayleigh, who thus obtains in place of Maxwell's (14) and (15) the following equations:—

$$\mu C = -\left(\alpha\mu\frac{dT}{dt} + \frac{2\alpha^2\mu^2}{1^2 \cdot 2^2}\cdot\frac{d^2T}{dt^2} + \dots + \frac{n\alpha^n\mu^n}{1^2 \cdot 2^2 \dots n^2}\cdot\frac{d^nT}{dt^n} + \dots\right), \;.\;.$$

$$AC - S = T + \alpha\mu\frac{dT}{dt} + \frac{\alpha^2\mu^2}{1^2 \cdot 2^2}\frac{d^2T}{dt^2} + \dots + \frac{\alpha^n\mu^n}{1^2 \cdot 2^2 \dots n^2}\frac{d^nT}{dt^n} + \dots\Big),$$

where α, equal to l/R, represents the conductivity (for steady currents) of unit length of the wire.

By writing

$$\phi(x) = 1 + x + \frac{x^2}{1^2 \cdot 2^2} + \dots + \frac{x^n}{1^2 \cdot 2^2 \dots n^2} + \dots \quad (4)$$

equations (2) and (3) are then transformed as follows:

$$\frac{dS}{dt} = A\frac{dC^*}{dt} - \phi\left(\alpha\mu\frac{d}{dt}\right)\cdot\frac{dT}{dt}, \quad (5)$$

$$C = -\alpha\phi'\left(\alpha\mu\frac{d}{dt}\right)\cdot\frac{dT}{dt}: \quad (6)$$

we have further

$$\frac{E}{l} = \frac{dS}{dt}. \quad (7)$$

Lord Rayleigh then applies equations (5), (6), and (7) to sustained periodic currents following the harmonic law, where all the functions are proportional to e^{ipt}, and obtains

$$E = R'C + ipL'C, \quad (8)$$

R' and L' denoting the effective resistance and inductance respectively to the currents in question. The values of R' and L' are expressed in the form of infinite series. For high frequencies, however, they are put also in a finite form, since, when p is very great, equation (4) reduces analytically to

$$\phi(x) = \frac{1}{2\sqrt{\pi}}\frac{e^{2\sqrt{x}}}{x^{\frac{1}{4}}}, \quad (9)$$

* AC is printed here in Phil. Mag., May 1886, p. 387; but appears to be a slip for $A\frac{dC}{dt}$.

so that

$$\frac{\phi(x)}{\phi'(x)} = x^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equivalent Resistance and Inductance for Oscillatory Discharges.—To effect the object of this paper we must now apply equations (5), (6), and (7) to the case of logarithmically-damped alternating currents where all the functions are proportional to $e^{(i-k)pt}$.

The value of E so obtained must then be separated into real and imaginary parts as in (8), and then, together with the imaginary quantities, must be collected a proportionate part of the real ones so as to exhibit the result in the form

$$\mathrm{E} = \mathrm{R''C} + (i-k)p\mathrm{L''C}. \quad . \quad . \quad . \quad . \quad (11)$$

The quantities denoted by R″ and L″ in this equation will then represent what may be called the *equivalent* resistance and inductance of length l of the wire to the damped periodic currents under discussion. For, the operand being now $e^{(i-k)pt}$, the time differentiator produces $(i-k)p$, and not ip simply as in equation (8) for the sustained harmonic currents.

Thus (5), (6), and (7), on elimination of S, l, and $\frac{d\mathrm{T}}{dt}$, give

$$\frac{\mathrm{E}}{\mathrm{RC}} = (i-k)p\alpha\mathrm{A} + \frac{\phi(ip\alpha\mu - kp\alpha\mu)}{\phi'(ip\alpha\mu - kp\alpha\mu)}. \quad . \quad . \quad (12)$$

Now we have

$$\frac{\phi(x)}{\phi'(x)} = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{48} - \frac{x^4}{180} + \ldots; \quad . \quad (13)$$

thus

$$\frac{\phi(ip\alpha\mu - kp\alpha\mu)}{\phi'(ip\alpha\mu - kp\alpha\mu)}$$

$$= 1 - \tfrac{1}{2}kp\alpha\mu + \frac{1-k^2}{12}p^2\alpha^2\mu^2 + \frac{k(3-k^2)}{48}p^3\alpha^3\mu^3 - \frac{1-6k^2+k^4}{180}p^4\alpha^4\mu^4 \ldots .$$

$$+ i\left\{\tfrac{1}{2}p\alpha\mu + \frac{k}{6}p^2\alpha^2\mu^2 - \frac{1-3k^2}{48}p^3\alpha^3\mu^3 - \frac{4k(1-k^2)}{180}p^4\alpha^4\mu^4 \ldots .\right\}. \quad (14)$$

Hence, substituting (14) in (12) and collecting the terms as in (11), we find that

$$\frac{\mathrm{R''}}{\mathrm{R}} = 1 + \frac{1+k^2}{12}p^2\alpha^2\mu^2 + \frac{k(1+k^2)}{24}p^3\alpha^3\mu^3 - \frac{1-2k^2-3k^4}{180}p^4\alpha^4\mu^4 \ldots, \quad (15)$$

and

$$\left.\begin{aligned}\frac{L''}{R} &= \alpha A + \tfrac{1}{2}\alpha\mu + \frac{k}{6}p\alpha^2\mu^2 - \frac{1-3k^2}{48}p^2\alpha^3\mu^3 - \frac{k(1-k^2)}{45}p^3\alpha^4\mu^4 \ldots\\ &\text{or}\\ L'' &= l\left[A + \mu\left(\tfrac{1}{2} + \frac{k}{6}p\alpha\mu - \frac{1-3k^2}{48}p^2\alpha^2\mu^2 - \frac{k(1-k^2)}{45}p^3\alpha^3\mu^3 \ldots\right)\right]\end{aligned}\right\}$$

Putting $k=0$ in these equations and denoting by single-dashed letters the corresponding values of the resistance and inductance, we have

$$\frac{R'}{R} = 1 + \frac{1}{12}p^2\alpha^2\mu^2 - \frac{1}{180}p^4\alpha^4\mu^4 \ldots, \quad . \quad . \quad (17)$$

and

$$L' = l\left[A + \mu\left(\tfrac{1}{2} - \frac{1}{48}p^2\alpha^2\mu^2 \ldots\right)\right], \quad . \quad . \quad (18)$$

which are Lord Rayleigh's well-known formulæ * for periodic currents of constant amplitude.

By taking the differences of the resistances and inductances with damping and without, we have at once

$$\frac{R''-R'}{R} = k^2p^2\alpha^2\mu^2 + \frac{k(1-k^2)}{24}p^3\alpha^3\mu^3 + \ldots \quad . \quad . \quad (19)$$

and

$$L''-L = l\mu\left(\frac{k}{6}p\alpha\mu + \frac{k^2}{6}p^2\alpha^2\mu^2 \ldots\right). \quad . \quad . \quad (20)$$

These show that if the frequency is such that a few terms sufficiently represent the value of the series, then *both resistance and inductance are increased by the damping.*

High-Frequency Discharges.—Passing now to cases where p is very great, as in the wave-trains in or induced by a Hertzian primary oscillator, we have from equation (10),

$$\frac{\phi(ip\alpha\mu - kp\alpha\mu)}{\phi'(ip\alpha\mu - kp\alpha\mu)} = \sqrt{(i-k)p\alpha\mu} = (p\alpha\mu s)^{\frac{1}{2}}\left(i\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right), \quad (21)$$

where $s = \sqrt{1+k^2}$ and $\cot\theta = k$.

* Equations (19) and (20), p. 387, *loc. cit.*

On substituting this value of ϕ/ϕ' in equation (12) and collecting as before, we obtain the solution sought, viz.:

$$\frac{E}{RC}=(\alpha\mu p s^3)^{\frac{1}{2}}\cos\frac{\theta}{2}+(i-k)p\left(\alpha A+\sqrt{\alpha\mu s/p}\cos\frac{\theta}{2}\right); \quad (22)$$

whence
$$\frac{R''}{R}=(\alpha\mu p s^3)^{\frac{1}{2}}\cos\frac{\theta}{2}, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and
$$\left.\begin{aligned}\frac{L''}{R}&=\alpha A+(\alpha\mu s/p)^{\frac{1}{2}}\cos\frac{\theta}{2},\\ \text{or}\quad L''&=l\left[A+\left(\frac{\mu s}{\alpha p}\right)^{\frac{1}{2}}\cos\frac{\theta}{2}\right].\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (24)$$

Discussion of the Results for High Frequencies.

On putting $k=0$, in equations (23) and (24), to reduce to the case of sustained simple harmonic waves, $s=1$, $\theta=\frac{\pi}{2}$; whence, denoting by single dashes these special values of R″ and L″, we obtain

$$\frac{R'}{R}=\sqrt{\tfrac{1}{2}\alpha\mu p}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and
$$L'=l\left\{A+\sqrt{\frac{\mu}{2\alpha p}}\right\}, \quad . \quad . \quad . \quad . \quad (26)$$

which are Lord Rayleigh's high-frequency formulæ*.

Referring again to equations (23) and (24), we see that for a given value of p, if k varies from 0 to ∞, the factor involving s increases without limit while that involving θ increases to unity. Hence, with increasing damping, it appears that R″ and L″ each increase also, while ever the equations remain applicable. Now an infinite value of k involves zero frequency†. And a certain large, though finite, value of k would prevent the frequency being classed as "high."

* Equations (26) and (27), p. 390, *loc. cit.*

† This follows from the fact that electric currents or waves generated by an oscillatory discharge may be represented by $e^{-kpt}\cos pt$, in which kp is finite, so k is infinite only when p is zero.

Dividing equation (23) by (25) gives

$$\frac{R''}{R'}=(2s^3)^{\frac{1}{2}}\cos\frac{\theta}{2}=K \text{ say.} \quad . \quad . \quad . \quad . \quad (27)$$

Thus, for a given value of k, the ratio R''/R' is independent of the frequency of the waves. It is therefore convenient to deal with K a function of k only, rather than with R''/R which is a function of p also.

Differentiating to k, we have

$$\frac{dK}{dk}=\frac{3k\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\sqrt{2s}}, \quad . \quad . \quad . \quad . \quad . \quad (28)$$

which is positive for all values of k from 0 to ∞, hence K increases continuously with k. For $k=0$, this becomes

$$\left(\frac{dK}{dk}\right)_{k=0}=\frac{\sin\frac{\pi}{4}}{\sqrt{2}}=\frac{1}{2}, \quad . \quad . \quad . \quad . \quad (29)$$

which assists in plotting K as a function of k.

Differentiating again, we obtain

$$\frac{d^2K}{dk^2}=\frac{1}{\sqrt{2^3s^5}}\left\{(7+3k^2)\cos\frac{\theta}{2}+2k\sin\frac{\theta}{2}\right\} \quad . \quad (30)$$

Since this expression is positive for all values of k from 0 to ∞, we see that K plotted as a function of k is a curve which is always convex to the axis of k. Thus the nature of R''/R' as a function of k is sufficiently determined.

Pairs of corresponding values of K and k for a few typical cases are shown in the accompanying table, and part of the curve coordinating them is given in fig. 1. It is not necessary to plot much of the curve, as only a small part of it can apply to any actual case. For, although k may have any positive value up to ∞, the high values of k, as already mentioned, correspond to low values of p and so exclude them from the application of the high-frequency formula.

Fig. 1.—Exhibiting graphically $K=R''/R'$ as a function of k, the damping factor.

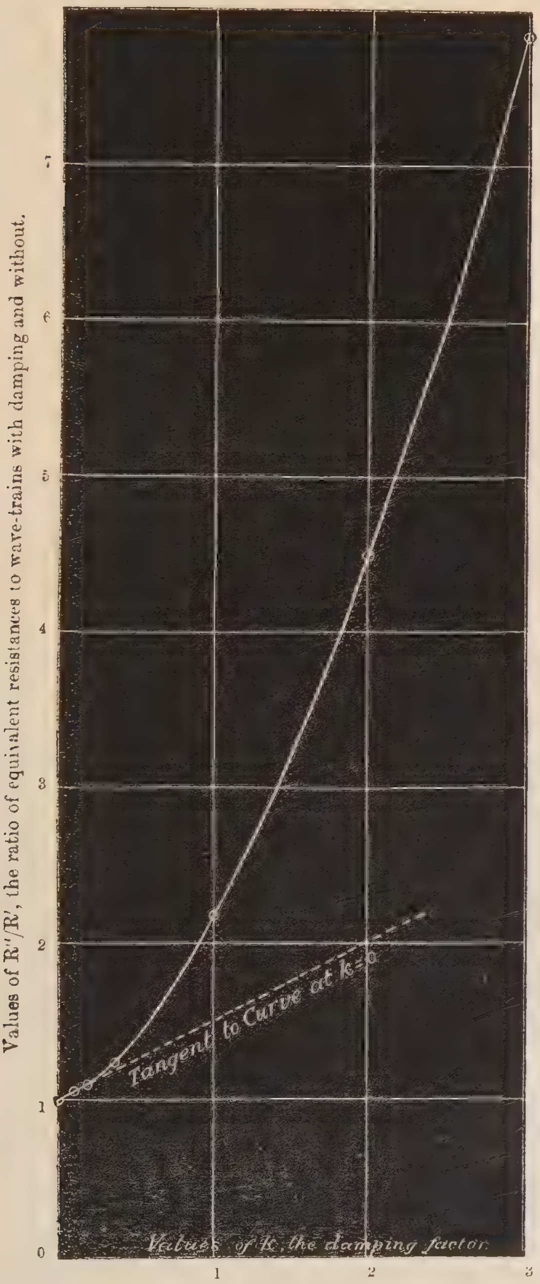


TABLE showing the values of $K=R''/R'$, the ratio of equivalent resistances to waves with damping and without.

Damping Factor, $k=\cot\theta$.	Subsidiary quantities involved.		Ratio of Resistances $K=R''/R'$.
	$\theta/2$.	$s^2=1+k^2$.	
0	45°	1	1
$\frac{1}{4\pi}=0{\cdot}0798$	42° 44′	1·006362	1·044 nearly
$\frac{3}{10\pi}=0{\cdot}0955$	42° 16′	1·00913	1·054 „
$\frac{1}{2\pi}=0{\cdot}1595$	40° 28′	1·02614	1·097 „
$\frac{1}{\pi}=0{\cdot}319$	36° 9′	1·1018	1·228 „
1	22° 30′	2	2·197 „
2	13° 17′	5	4·602 „
3	9° 13′	10	7·85 „

Figure 2 shows the form of a wave-train for which $k=1$ and $K=2{\cdot}197$. That is to say, in this extreme case where all the functions vary as $e^{(i-1)pt}$, and the wave-train passing a given point of the wire is accordingly represented by $e^{-pt}\cos pt$, then the equivalent resistance is 2·197 times that which would obtain for simple harmonic waves uniformly sustained and of the same frequency.

Figure 3 represents the form of the wave-trains generated and used in some recent experiments on attenuation*. In this case the value of k was approximately $\frac{3}{10\pi}$, or the logarithmic decrement per wave $=2\pi k=0{\cdot}6$, and the corresponding value of K, the ratio of R''/R', is 1·054. Now in the experiments just referred to the frequency was 35×10^6 per second, and R'/R became 31·6. Hence R''/R has the

* "Attenuation of Electric Waves along a Line of Negligible Leakage," Phil. Mag. Sept. 1898, pp. 296–305.

Fig. 2.—Instantaneous Form of Wave-train for $k=1$, whence $R''/R'=2{\cdot}197$.

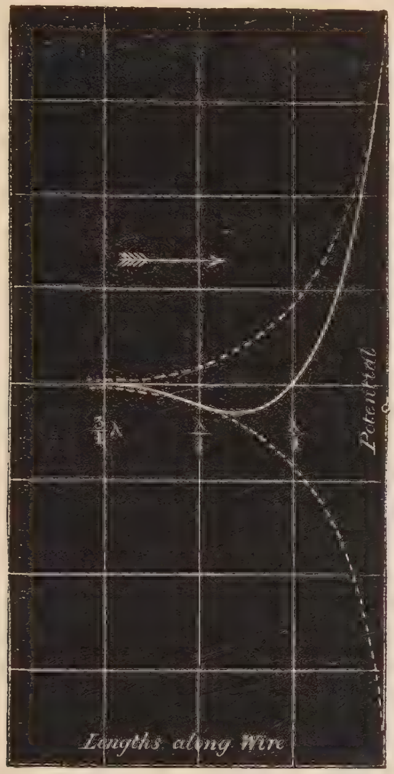
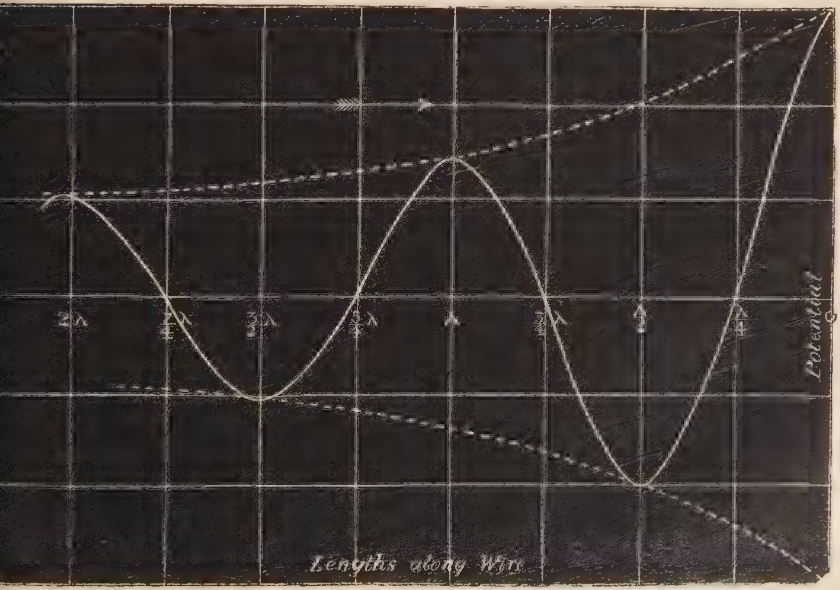


Fig. 3.—Instantaneous Form of Wave-Train for $k={\cdot}3/\pi$, whence $R''/R'=1{\cdot}054$.



value $31 \cdot 6 \times 1 \cdot 054 = 33 \cdot 3$ nearly. Thus, writing $e^{-R''x/2Lv}$ for the attenuator of the waves along the wires instead of * $e^{-R'x/2Lv}$ increases the index by about five and a half per cent., and so brings it by that amount nearer to the value determined experimentally.

Univ. Coll., Nottingham,
Nov. 29, 1898.

Discussion.

Mr. Oliver Heaviside (communicated).—I have in another way obtained the same number $R''/R' = 1 \cdot 054$, showing that the effective resistance given by Lord Rayleigh's formula should be increased 5·4 per cent. when the impressed vibrations are damped to the extent Dr. Barton found in his experiments. This R'' is what should take the place of the resistance per unit length of circuit in the factor $\epsilon^{-Rx/2Lv}$, showing the attenuation in transit of waves along the circuit. This R'', however, underestimates the resistance, because the formula is not valid right up to the wave front, but is what is tended to later on. But it does not seem likely that a complete reckoning of the effect of the variable resistance of the wires will make up the deficit of theory from observation. Some other causes of increased attenuation have been suggested. In addition it is even possible that the conductivity of copper to vibrations millions per second may be less than with steady currents, and that the voltage at the beginning of the wave-train may be large enough to cause some leakage.

Both R'' and L'' go up to infinity with infinite increase in the damping. But the change in the inductance is quite insensible in the experiment. The inductance is sensibly that of the dielectric.

As regards the meaning of R'' and L'', they differ somewhat from R' and L'. When we reduce the equation of voltage $E = ZC$ to the form $E = \left(R'' + L''\frac{d}{dt}\right)C$, both E and C being of the type $\epsilon^{-at}\sin(nt+\theta)$, we have the activity

* See Equation (2) p. 301, Phil. Mag. Sept. 1898.

equation

$$EC = R''C^2 + \frac{d}{dt}\tfrac{1}{2}L''C^2,$$

and also

$$EC = Q + \frac{dT}{dt},$$

where Q is the waste and T the magnetic energy. But the mean Q is not the same as the mean $R''C^2$, nor the mean T the same as the mean $\frac{1}{2}L''C^2$, save when $a=0$, or the vibrations are undamped. It is true, however (as I have investigated), that the mean $Q\epsilon^{2at}$ and $T\epsilon^{2at}$ are the same as the mean $R''C^2\epsilon^{2at}$ and $\frac{1}{2}L''C^2\epsilon^{2at}$, so that if ϵ^{2at} does not change sensibly in a period, R'' and L'' are sensibly the effective resistance and inductance in the same sense as R' and L'.

XL. *Exhibition and Description of Wehnelt's Current-Interrupter.* *By* A. A. Campbell Swinton.*

[Abstract.]

A glass cell contains a large cylindrical negative electrode of lead, and a small positive electrode consisting of a platinum wire about $\frac{1}{4}$ inch in length, in a solution of 1 part sulphuric acid to about 5 parts water. The platinum wire may project from the top of the shorter arm of a J-shaped ebonite tube, so that it can point upwards, immersed in the solution. Or it may be fused into a similar glass tube; but glass is apt to crack in the subsequent heating. Wehnelt's interrupter replaces the make-and-break apparatus of an induction-coil; it also replaces the ordinary condenser of that apparatus. In its present form it requires rather a strong current. The resulting spark at the secondary terminals differs in character from the ordinary spark of an induction-coil; it is almost unidirectional, and in air takes a Λ-form—bright, continuous, and inverted—somewhat like a pair of flaming swords rapidly

* Read March 10, 1899.

crossing and re-crossing one another at their points. By blowing upon the discharge it breaks up, and then more nearly resembles the customary discharge of a coil. The sound emitted by the spark has a pitch that varies with the conditions of the circuit. As the self-induction of the circuit is diminished, the spark-pitch rises ; it becomes infinite when tke self-induction vanishes, *i. e.*, the Wehnelt interrupter will not work in a circuit devoid of self-induction. As the applied potential difference diminishes, the spark-pitch diminishes. In the author's experiments 25 volts was the minimum primary voltage at which the apparatus would work. The spark-pitch also varies with the length of the platinum-wire electrode in the solution. If the circuit is closed by slowly dipping this electrode into the solution, the apparatus will not work ; the wire should be dipped in before closing the circuit, or at any rate immersed with considerable rapidity. After working for about a quarter of an hour the action often ceases ; this fatigue effect is not due to heating of the solution, for it is not obviated by keeping the temperature constant by a water-bath. It is supposed that the oxygen generated at the platinum electrode forms a more or less insulating film which interrupts the current until absorbed by the surrounding water. The fact that oxygen is more easily absorbed than hydrogen, may explain why it is necessary to connect the platinum electrode to the positive pole of the battery or dynamo. When the platinum electrode is dipped gradually into the solution, the wire gets red-hot, and the interruptions do not take place. Again, when the apparatus stops from fatigue the platinum gets red-hot. The action is further complicated by a series of small explosions, and by the formation of a kind of intermittent electric arc at the platinum electrode. The coil exhibited was connected to the 100-volt electric light mains at Burlington House ; in this case the potential difference at the terminals of the primary was 30 volts, and that across the interrupter 150 volts—a total of 180 volts, showing the effect of self-induction. For Röntgen-ray work the apparatus would be very effective, but unfortunately the sparks produce great heating, so that the terminals of the tubes are liable to be melted.

The author suggests that, as the sparks are more nearly continuous than ordinary discharges if used for producing Hertz waves, the trains of waves would follow one another at shorter intervals than those from the sparks at present employed. In fact it might eventually be possible in this way to obtain the almost absolutely continuous trains of waves that are necessary for proper syntony in wireless telegraphy.

Discussion.

The President asked whether the self-induction of the primary coil was not sufficient of itself to form the induction factor in the impedance necessary for perfect working. He would like to know how the apparatus behaved when an alternating current was used. Did the secondary coil become damaged by over-heating? Did reversal of the current assist the recovery from the fatigued condition of the apparatus? The natural period of the circuit probably depended upon its capacity and its self-induction. There must undoubtedly be some capacity at the surface of the platinum electrode in the liquid; this capacity might act with the auxiliary self-induction and the self-induction of the rest of the circuit in the orthodox way, and possibly there was automatic adjustment of resonance to the frequency of the interruptions, for instance, by variations of the capacity at the electrode. The heating effect when a wire was made to close a circuit with a liquid was discovered many years ago.

Prof. G. M. Minchin thought that the usefulness of the apparatus would be greatly increased if it could be made to work with less current. He had himself succeeded with an applied E.M.F. of 12 volts, but not with 10 volts. As a tentative experiment he had used a horizontal lead plate—with disastrous effect, for the apparatus went suddenly to pieces. Explosions were frequently obtained, but they were not attended with much real danger. In a later and safer apparatus he used a platinum wire about $\frac{3}{4}$ inch long, projecting from a glass tube, around which the lead plate was bent. There appeared to be a definite depth of immersion of this wire,

at which the apparatus worked with minimum current. In his apparatus this critical position was when half the wire was below the surface of the liquid, the other half projecting into the air. He attributed the fatigue to the presence of gas about the electrodes, for he observed that a mechanical tap to the base of the apparatus restored the working condition.

Mr. Rollo Appleyard pointed out that the improved result at half immersion observed by Prof. Minchin, taken together with the phenomena described by Mr. Campbell Swinton as to the effect of dipping the electrodes into the solution, suggested that the liquid immediately around the submerged part of the wire was at some instants in the spheroidal state. The breaking-down of the spheroidal state would be facilitated by heat lost by the immersed part to the non-immersed part of the wire. The capacity for heat of the non-immersed part, and the degree of roughness or smoothness of the immersed part, would thus appear as factors in the explanation. No doubt the evolved gases were the primary cause of the interruption of current, but the wire having once become red-hot the spheroidal condition would introduce a further cause of electrical separation between the wire and the liquid.

Prof. C. V. Boys asked whether it was the liquid or the electrodes that became fatigued. Experiments should be made to determine the effect of variations in the hydrostatic pressure around the platinum electrode.

Mr. T. H. Blakesley said that the rise of potential at the terminals of the interrupter proved that the arrangement possessed capacity. Such a rise of potential could not occur without there being capacity any more than it could without self-induction.

Dr. D. K. Morris described experiments he had made with a Wehnelt interrupter, using a 1 kilowatt transformer with a transformation ratio of 4 to 5, intended for 10 amperes at 100 volts. The anode of the interrupter was designed to have an adjustable surface to correspond with the load on the secondary—a platinum wire at the end of a copper wire could be projected more or less through the drawn-out lower end of a glass tube containing oil. The best results with the

interrupter were obtained with about 45 volts on the primary circuit. At this pressure, an average current of 1 ampere sufficed to give 125 (alternating) volts very steadily on the secondary, as measured by an electrostatic instrument. The "no-load" loss was thus only 45 watts. The secondary could then be loaded up with lamps, provided that the exposed surface of platinum wire was proportionately increased. The energy delivered to the lamps, however, was not at any load much greater than 45 per cent. of that taken from the mains. By connecting the interrupter with a condenser of ½ microfarad capacity, the efficiency at small loads was increased to nearly 60 per cent. He had observed that the fatigue of the interrupter could be temporarily remedied by reversing the current.

Mr. C. E. S. PHILLIPS asked whether Mr. Campbell Swinton had tried other liquids than dilute sulphuric acid. So far as his own experiments went, he had only obtained good results with that electrolyte.

Mr. CAMPBELL SWINTON, in reply, said that with the apparatus arranged in a simple circuit, an alternating current applied to the primary of an induction-coil through a Wehnelt interrupter produced only about half the effect of the corresponding direct current—apparently only the currents in one direction got through. But if two interrupters were connected in parallel circuits, it was possible so to arrange them that one took one half and the other the second half of the alternations. It might therefore be possible to design an induction-coil with two primary windings to correspond to the two interrupters, so as to give an additive effect. The ten-inch induction-coil he had used had suffered no damage from the currents employed in the experiments exhibited; though he had used as much as 20 primary amperes, there was extremely little heating of the secondary. He could not with his apparatus restore the working condition by any mechanical disturbance of the interrupter when once the fatigue effects had set in to any marked extent. He had tried other liquids in place of the dilute sulphuric acid. Whilst a saturated solution of potassic bichromate gave fair results, strong hydrochloric acid would not work at all.

The PRESIDENT said he did not altogether agree with Mr. Campbell Swinton's remarks as to the chances of being able to maintain electric oscillations, and so of improving Hertzian telegraphy, by the use of these interrupters. The rate of interruption with this apparatus was something like 1000 per second, but the vibrations corresponding to Hertz waves were of the order 100,000 per second. The wave-trains from oscillations excited by the new interrupter would still be series of damped vibrations; the amplitudes would not be maintained. It might be advantageous to have sparks following one another so rapidly, but he doubted it. For Hertzian telegraphy, the spark at the oscillator should 'crackle'; to produce the best effect, the air about the oscillator should be in a non-conducting condition.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1898-99.

February 26th, 1898.

Meeting held in the Physical Laboratory of Eton College by invitation of the Rev. T. C. Porter.

Shelford Bidwell, Esq., President, in the Chair.

The following were elected Fellows of the Society:—
Messrs. W. G. A. Bond and F. M. Saxelby.

Mr. Porter read the following communications:—

(1) Winter observations on the Shadow of the Peak of Tenerife, with a new method for measuring approximately the Diameter of the Earth.

(2) A new Theory of Geysers.

(3) A method of viewing Lantern Projections in Stereoscopic Relief.

March 11th, 1898.

Shelford Bidwell, Esq., President, in the Chair.

The following were elected Fellows of the Society:—
Messrs. J. M. Davidson, C. V. Drysdale, and J. H. Howell.

Prof. J. D. Everett read a paper "On Dynamical Illustrations of certain Optical Phenomena."

Mr. R. A. Lehfeldt read a paper "On the Properties of Liquid Mixtures."

March 25th, 1898.

Shelford Bidwell, Esq., President, in the Chair.

The following were elected Fellows of the Society:—

Messrs. H. Darwin, C. F. Green, and J. H. Thomson.

Mr. A. A. C. Swinton read a paper "On the Circulation of the Residual Gaseous Matter in a Crookes Tube."

Mr. A. Stansfield read a paper "On some Improvements in the Roberts-Austen Recording Pyrometer, and Notes on Thermoelectric Pyrometry."

April 22nd, 1898.

Shelford Bidwell, Esq., President, in the Chair.

The following were elected Fellows of the Society:—

Prof. J. A. Ewing and Mr. S. G. Starling.

The Rev. T. C. Porter communicated a paper "On a Method of viewing Newton's Rings."

Prof. S. P. Thompson exhibited a model Triphase Generator and Motor."

May 13th, 1898.

Shelford Bidwell, Esq., President, in the Chair.

The following were elected Fellows of the Society:—

Messrs. L. Gaster and J. Sampson.

Prof. Ayrton and Mr. Mather read a paper "On Galvanometers.—Part II."

May 27th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society:—
Messrs. F. W. CARTER and J. K. MOORE.

Mr. EDSER and Mr. BUTLER read a paper "On a Simple Interference Method of Reducing Prismatic Spectra."

Mr. A. A. C. SWINTON read a paper "On some further Experiments on the Circulation of the Residual Gaseous Matter in Crookes Tubes."

June 10th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

Prof. S. P. THOMPSON exhibited and described a Model illustrating Prof. Max Meyer's new Theory of Audition.

Mr. BARTON read a paper "On the Attenuation of Electric Waves along a Line of Negligible Leakage."

Mr. A. GRIFFITHS read a paper "On Diffusive Convection."

June 24th, 1898.

WALTER BAILY, Esq., Vice-President, in the Chair.

The following was elected a Fellow of the Society:—
Prof. EUGENIO SEMMOLA.

Prof. CARUS WILSON exhibited an Apparatus illustrating the action of Two Coupled Electric Motors.

Mr. J. QUICK exhibited Weedon's Expansion of Solids Apparatus.

Dr. F. G. DONNAN communicated a paper "On the Theory of the Hall Effect in a Binary Electrolyte."

October 28th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

Mr. W. R. PIDGEON read a paper on "An Influence Machine."

Prof. S. P. THOMPSON showed an Experiment on the Magneto-optic Phenomenon discovered by Righi.

Mr. A. CAMPBELL read a paper "On the Magnetic Fluxes in Meters and other Electrical Instruments."

November 11th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The discussion on Mr. CAMPBELL's paper read at the last meeting was continued.

Prof. W. B. MORTON communicated a paper "On the Propagation of Damped Electrical Oscillations along Parallel Wires."

November 25th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society :—

Messrs. J. B. BAYLIS, R. W. FORSYTH, and WILSON NOBLE.

Mr. R. A. LEHFELDT read a paper "On the Properties of Liquid Mixtures."

Mr. L. N. G. FILON read a paper "On certain Diffraction Fringes as applied to Micrometric Observations."

December 9th, 1898.

SHELFORD BIDWELL, Esq., President, in the Chair.

The following were elected Fellows of the Society :—

Prof. H. L. CALLENDAR and Mr. R. S. WHIPPLE.

Dr. C. CHREE read a paper "On Longitudinal Vibrations in Solid and Hollow Cylinders."

Mr. ROSE-INNES and Prof. SYDNEY YOUNG read a paper "On the Thermal Properties of Normal Pentane."

January 27th, 1899.

G. GRIFFITH, Esq., Vice-President, in the Chair.

Dr. BARTON read a paper "On the Equivalent Resistance and Inductance of a Wire to an Oscillatory Discharge."

Mr. APPLEYARD exhibited a Dephlegmator and a Temperature Tell-tale.

Mr. LITTLEWOOD read a paper "On the Volume-changes accompanying Solution."

Annual General Meeting.

February 10th, 1899.

Shelford Bidwell, Esq., President, in the Chair.

The following Report of the Council was read by the Secretary:—

Since the last Annual General Meeting of the Society thirteen meetings have been held. One of these, on February 26th, 1898, took place in the Science Schools at Eton, on the kind invitation of the Rev. T. C. Porter, who provided some interesting demonstrations and made the Fellows of the Society welcome. All the other meetings were held in the rooms of the Chemical Society at Burlington House.

Twenty-one new Fellows have been elected into the Society; and the Council regrets to report that death has deprived the Society of a member of Council, Mr. Latimer Clark, and of Sir J. N. Douglass, Dr. John Hopkinson, Dr. J. E. Myers, the Rev. Bartholomew Price, Mr. Henry Perigal, and Dr. Eugen Obach. Short memoirs of some of these will be published in the Society's 'Proceedings.' There have been three resignations.

The Joint Committee of the Society and of the Institution of Electrical Engineers, which commenced its labours in January 1898, has published the first volume of 'Science Abstracts' in twelve monthly parts, being a continuation and expansion of the 'Abstracts of Papers on Physical Science' hitherto published by the Society alone. A much larger number of papers has been included, and the work has been very thoroughly done. Your Council is of opinion that it fulfils a want and is greatly appreciated. The expense has been considerable, and the estimate originally made has been somewhat exceeded. Your Council has, however, felt that this expense has been justified by the results, and has entered into arrangements for continuing the work of the Joint Committee.

It is confidently hoped that it will be possible to arrange for the permanent production of 'Science Abstracts' on a scale at least as extensive as the present.

The Report of the Council was adopted.

The Report of the Treasurer was read and adopted.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows :—

President.—Prof. Oliver J. Lodge, D.Sc., F.R.S.

Vice-Presidents who have filled the Office of President.—Dr. J. H. Gladstone, F.R.S.; Prof. G. C. Foster, F.R.S.; Prof. W. G. Adams, M.A., F.R.S.; Lord Kelvin, D.C.L., LL.D., F.R.S.; Prof. R. B. Clifton, M.A., F.R.S.; Prof. A. W. Reinold, M.A., F.R.S.; Prof. W. E. Ayrton, F.R.S.; Prof. G. F. Fitzgerald, M.A., F.R.S.; Prof. A. W. Rücker, M.A., F.R.S.; Capt. W. de W. Abney, R.E., C.B., D.C.L., F.R.S.; Shelford Bidwell, M.A., LL.B., F.R.S.

Vice-Presidents.—T. H. Blakesley, M.A.; C. Vernon Boys, F.R.S.; G. Griffith, M.A.; Prof. J. Perry, D.Sc., F.R.S.

Secretaries.—W. Watson, B.Sc.; H. M. Elder, M.A.

Foreign Secretary.—Prof. S. P. Thompson, D.Sc., F.R.S.

Treasurer.—Dr. E. Atkinson.

Librarian.—W. Watson, B.Sc.

Other Members of Council.—Prof. H. E. Armstrong, D.Sc., F.R.S.; Walter Baily, M.A.; R. E. Crompton; Prof. J. D. Everett, D.C.L., F.R.S.; Prof. A. Gray, LL.D., F.R.S.; E. H. Griffiths, M.A., F.R.S.; Prof. J. Viriamu Jones, M.A., F.R.S.; S. Lupton, M.A.; Prof. G. M. Minchin, M.A., F.R.S.; J. Walker, M.A.

Votes of thanks were passed to the Auditors, the Officers and Council, and to the Chemical Society.

The newly-elected President (Prof. Lodge) then took the Chair and delivered an Address, which is printed on pp. 343–386 of the current number of the 'Proceedings.'

The Treasurer in Account with the Physical Society, from January 1st, 1898, to December 31st. 1898.

Dr.	£ s. d.	£ s. d.
Jan. 1, 1897. Balance in the Bank		160 5 2
Entrance-Fees	21 0 0	
Subscriptions for 1897	4 4 0	
„ 1898	346 5 0	
Extra Subscriptions for 1898	33 12 0	
Subscriptions for 1899	2 2 0	
		407 3 0
Life Compositions		63 0 0
Completion of Voluntary Composition	5 15 0	
Arrears of incomplete Subscriptions	2 4 0	
		7 19 0
Dividends		77 6 8
Sales of publications		48 3 0
Grant from the British Association	100 0 0	
„ „ Institution of Civil Engineers	100 0 0	
		200 0 0
Balance due to the Treasurer		13 14 5
		£977 11 3

Cr.	£ s. d.	£ s. d.
Balance due to the Treasurer, Jan. 1, 1898		2 17 2
Abstractors for 1897		29 16 0
Taylor and Francis, for printing:—		
Abstracts for 1897	36 5 6	
Proceedings	88 14 0	
Separate Copies	14 11 0	
Miscellaneous Printing	35 8 0	
		174 18 6
Asiatic Society, annual rent for name plate		2 2 0
Attendance		2 3 0
Binding		1 19 3
Cheque Book and Commission on Cheques		8 10
Reporting		22 17 5
Petty Cash:—		
Mr. Boys	12 7	
Mr. Elder	2 2 0	
Mr. Watson	15 0	
Dr. Atkinson	2 5 0	
		5 14 7
Mr. Swinburne: Science Abstracts	250 0 0	
Return of Grant to the Joint Committee	50 0 0	
„ „ „	50 0 0	
		350 0 0
Balance in the Bank		384 14 6
		£977 11 3

EDMUND ATKINSON, *Treasurer.*

Audited and found correct.

ALFRED W. PORTER,
WILLIAM BARLOW, } *Honorary Auditors.*

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31, 1898.

Assets.

	£	s.	d.	£	s.	d.
Subscriptions due, estimated				40	0	0
£400 Furness 4 per cent. Debenture Stock	456	0	0			
£1000 Midland 4 per cent. Preference Stock	1194	0	0			
£200 Metropolitan Board of Works $3\frac{1}{2}$ per cent. Stock	217	0	0			
£270 Lancaster Corporation 3 per cent. Stock	227	6	8			
£254 2*s.* 9*d.* New South Wales $3\frac{1}{2}$ per cent. Inscribed Stock	249	7	5			
				2343	14	1
Balance in the Bank				384	14	6
Guarantee Fund, on Deposit				424	12	2
				£3193	0	9

Liabilities.

	£	s.	d.
Balance due to the Treasurer	13	14	5
Rent due to the Chemical Society	20	9	0
Abstracts: due to the Joint Committee	70	0	0
Printers	43	14	3
Balance	3045	3	1
	£3193	0	9

EDMUND ATKINSON, *Treasurer.*

Securities examined and found correct.

ALFRED W. PORTER, } *Honorary Auditors.*
WILLIAM BARLOW, }

The Meeting then resolved itself into an Ordinary Science Meeting, and the following were elected Fellows of the Society :—

Messrs. David Woolacot and J. Donaldson.

Mr. B. Davis communicated a paper on "An Ampere-meter and a Voltmeter with a long Scale."

Dr. John Hopkinson, F.R.S., was born on July 27, 1849, and received his early scientific training at Owens College, Manchester, and Trinity College, Cambridge. He was Senior Wrangler and first Smith's Prizeman, as well as a Whitworth Scholar and a D.Sc. of London University. Shortly after leaving Cambridge he joined the firm of Messrs. Chance Brothers & Company, and while with them introduced many and important improvements in lighthouse illumination. He also conducted some important researches on the electrical properties of glass. In 1879 the first of the long series of papers in connection with electric lighting and engineering, which has rendered his name famous, was published. In 1886, in conjunction with his brother Edward Hopkinson, was published the paper on the general theory of the magnetic circuit, which had such a revolutionizing influence on the design of dynamo machinery.

He was a Fellow of the Royal Society, and was twice President of the Institution of Electrical Engineers. He became a Fellow of the Physical Society in 1878.

He was an expert climber, and on August 27, 1898, together with one of his sons and two daughters, he set out to climb the Petite Dent de Veisivi. As they did not return at nightfall, search parties were sent out, and the four bodies were found at the base of a cliff, having fallen from a height of about 500 feet. By this sad accident the scientific world lost a leader whose place cannot easily be filled.

Mr. Latimer Clark, F.R.S., was born at Great Marlow in 1822. He was engaged in a chemical industry in Dublin, when the rapid development of railways tempted him to serve as a pupil with his brother, Mr. Edwin Clark, on the Chester and Holyhead Railway. He assisted in the famous experiments which preceded the construction of the Britannia tubular bridge. The bridge was of so novel a kind that it could not be trusted to a Contractor, so that

from 1848 to 1850 Mr. Clark had an experience of experimenting, calculating, designing, and of directing workmen. His bent towards electrical experimenting caused him to be asked to join the staff of the Electric Telegraph Company, of which he was Engineer-in-chief from 1854 to 1861, and Consulting Engineer till 1870, when the Government took charge of telegraphic work. It is claimed for him that he was the first to suggest and carry out (in 1854) the pneumatic conveyance of parcels and letters. In 1854 he gave a lecture at the Royal Institution on the results of his experiments on submarine and subterranean electric conductors. In 1861 his paper read before the British Association introduced the idea of the use of definite practical electrical units and standards, with such names as Ohm, Farad, and Volt. In 1873 he brought before the Royal Society the results of his elaborate, costly, and long investigation which led to the use of the Clark's cell as our standard of Electromotive Force. He was a good practical astronomer; a photographer; a great collector of books on Electricity and Magnetism; a writer of a number of very useful handbooks; an inventor of many things now in common use; an engineer who constructed floating docks and large hydraulic contrivances; a manufacturer of electrical machinery and apparatus; a layer of submarine cables. He was President of the Institution of Electrical Engineers in 1875, and became a Fellow of the Royal Society in 1889. Mr. Clark was elected a Fellow of the Physical Society during its first Session, and at the time of his death was on the Council.

The Rev. BARTHOLOMEW PRICE, D.D., Master of Pembroke College, Oxford, died on December 29th, 1898, in the 81st year of his age.

He was widely known by his exhaustive treatises on the Calculus, and was more noted perhaps for his business and financial ability; he remained active and vigorous until he was eighty, and by his kindliness made for himself a large circle of friends.

Beyond this, however, there is a side to his career which is of special interest to our Society: at a very early date his sagacity was recognized at Oxford: his clear and temperate views gained for him such influence that on important occasions the leaders of all parties would seek his counsel, and what was to him right and politic did not generally meet with opposition. This power was exercised in the cause of liberal education. In all that has been done for the study of Natural Science at Oxford during the last

forty years no one has taken a greater part than the late Professor Price. Much still remains to be done, but the importance of his work may be seen, when it is realized at what point it was necessary to begin. The regulation, that some knowledge of Arithmetic must be shown by all students entering the University of Oxford, was proposed and carried through by our late Fellow. He was always working for progress, but was never ruffled by the opposition of the mediæval spirit: without his tact and good temper the progress of learning, as we understand it, might have been delayed for many years in Oxford.

It may be of interest to note that he was the President of Section A, at the famous meeting of the British Association at Oxford in 1860, when Bishop Wilberforce, in the one incautious moment of his life, attacked Professor Huxley.

Mr. Henry Perigal was elected a member of the Physical Society during the first Session of the Society. He died at the advanced age of ninety-seven years, and till within a short period of his death was a constant attendant at the meetings of the Society. He was Treasurer of the Royal Meteorological Society and a Fellow of the Royal Astronomical Society, and a member of several other scientific bodies.

Dr. Eugen F. A. Obach, F.I.C., F.C.S., was elected a Fellow of the Society in May 1880. He was a Doctor of Philosophy of Heidelberg. He was in the employ of Messrs. Siemens Bros. & Co., of Woolwich, and devoted himself to the study of guttapercha and indiarubber. He delivered the Cantor Lectures on these substances at the Society of Arts in 1898. He died at the early age of forty-six.

Sir James Nicholas Douglass, F.R.S , became a Fellow of the Society in 1889. He was many years Engineer-in-chief to the Hon. Corporation of Trinity House, and built several lighthouses. He also made several improvements in the illuminating apparatus for lighthouses and lightships.

CONTENTS.

Proceedings at the Meetings of the Physical Society of London, Session 1898–99.